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# The organisation. Analytical approaches regarding the organization

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***Abstract:*** The first part of the paper contains some results regarding the inverse image of the Smarandache function S, using the calculation algorithm for the function S, defined in [6]. Thus, the decomposition of the set of natural numbers into equivalence classes and some properties of these classes is presented, in order to make a connection with the cryptographic protection of the financial information, through the RSA cryptosystems.

***Key words:*** divisibility of natural numbers, division of modulo m, equivalence classes, algorithm, fixed point of an application, the preimage of a function, RSA cryptosystems.

***JEL Classification: C5, C6.***

## 1. Introduction

The development of mathematical concepts validates the apparatus of scientific research in various branches, with applicative connotations, insofar as the results can influence the sustainability of environments, including economic ones.

The numerical function **S** of Fl. Smarandache can be defined as follows [2], [7], [8]:

$$\mathbf{S}:\mathbb{N}^* \to \mathbb{N}^*,\ \mathbf{S}(n)=\min\{x \in \mathbb{N}^* | n | x!\}.$$

From [6] it is known that $\mathbf{S}(a) = \propto \cdot\, p$, if the number *a* has the canonical decomposition $a=p^{\propto}$, where $p$ is the prime natural number and $1 \le \propto \le p$. If $p$ is prime, $\mathbf{S}(p) = p$, i.e. $p$ is a fixed point of the function **S**. Moreover, it is found that the set of fixed points for **S** is Prim∪ {1,4}, where Prim is the set of prime natural numbers.

Known Legendre's formula [3], [4], [9], [10] is:

$$exp_{n!}(p) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots,$$

where, $p$ is a prime natural number, $n \in \mathbb{N}^*$ and $exp_{n!}(p)$ means the exponent of the greatest power at which $p$ appears in the decomposition of $n!$. A connection was established between the **S** function and Legendre's formula [6], after the following function was defined:

$$g:\mathbb{N}^* \times \text{Prim} \to \mathbb{N},\quad g(n,p) = \begin{cases} exp_{n!}(p), & n \ge p \\ 0, & 1 \le n < p \end{cases}.$$

Thus, if $(n, p) \in \mathbb{N}^* \times$ Prim, then:

$$\mathbf{S}\left(p^{g(n,p)}\right) \leq n,$$

equality being achieved when $p^{g(n,p)} \nmid (n-1)!$.

For $a \in \mathbb{N}^*$ the following representation is called **the canonical decomposition** or **factorization** of $a$:

$$a = p_1^{\alpha_1} p_2^{\alpha_2} \cdot \ldots \cdot p_k^{\alpha_k},$$

where $k \in \mathbb{N}^*$, $\alpha_1, \ldots, \alpha_k \in \mathbb{N}^*$ and $1 < p_1 < \cdots < p_i < \cdots < p_k$ are prime numbers [4].
If we determined $\mathbf{S}(p_i^{\alpha_i})$ for any $i = \overline{1,k}$ , then [6]:

$$\mathbf{S}(a) = \max\left\{\mathbf{S}(p_i^{\alpha_i}) \middle| i = \overline{1,k}\right\}.$$

According to [5], **the extended sum** of the digits of a natural number $a$ in base ten, denoted by $\bar{s}(a)$, is equal to the sum of the digits of $a$ if it is at most 9 or otherwise, the sum of the digits of the result of the first sum, then the sum of the result of the next sum and so on until the sum of the digits is at most nine. So the extended sum of the digits of a number in base 10 is between 0 and 9, that is:

$$\bar{s}: \mathbb{N} \to \{0,1, \ldots, 9\}.$$

For example, $\bar{s}$ (177788) = 2, because we obtain: $1 + 3 \cdot 7 + 2 \cdot 8 = 38$ and $38 > 9$, then $3 + 8 = 11$ and $11 > 9$, then $1 + 1 = 2 < 9$ , stop.
For any $a, b \in \mathbb{N}^*$, in base 10, the following relations are true: i) $\bar{s}(a) = s(a) \bmod 9$, if $s(a) \neq \mathcal{M}9$, where $\mathcal{M}9$ means a multiple of 9 şi $\bar{s}(a) = 9$ if $s(a) = \mathcal{M}9$; ii) $\bar{s}\left(\bar{s}(a)\right) = \bar{s}(a)$; iii) $\bar{s}(\bar{s}(a) + \mathcal{M}9) = \bar{s}(a)$; iv) $\bar{s}(a+b) = \bar{s}\left(\bar{s}(a) + \bar{s}(b)\right)$; v) $\bar{s}(a \cdot b) = \bar{s}(\bar{s}(a) \cdot \bar{s}(b))$; vi) $\bar{s}(a^n) = \bar{s}\left(\left(\bar{s}(a)\right)^n\right)$ .

## 2. The inverse image corresponding to the function S

Since $n! \mid n!$, for any $n \in \mathbb{N}^*$ and $n \neq 1$ can be written $\mathbf{S}(n!) = n$, because $n! \nmid (n-1)!$. In addition, $\mathbf{S}(1!) = 1$. Thus, the Smarandache function is surjective, i.e. the set $\{x \in \mathbb{N}^* | \mathbf{S}(x) = n\}$ is non empty for any $n \in \mathbb{N}^*$. To $\mathbf{S}$ is attached the inverse image $S^-$, where:

$$S^-: \mathcal{P}(\mathbb{N}^*) \to \mathcal{P}(\mathbb{N}^*).$$

Then the canonical surjection, denoted by $S_-$, defined as follows, makes sense:

$$S_-: \mathbb{N}^* \to \mathcal{P}(\mathbb{N}^*), \text{ where } S_-(n) = \{x \in \mathbb{N}^* | \mathbf{S}(x) = n\}.$$

Because $\mathbf{S}(n!) = n$, it results $n! \in S_-(n)$ for any $n \in \mathbb{N}^*$.

*Examples*.

$S_-(1) = \{1\}$, for $\mathbf{S}(1) = 1$; so $1! \in S_-(1)$; $S_-(2) = \{2\}$, for $\mathbf{S}(2) = \{2\}$; so $2! \in S_-(2)$.

$S_-(3) = \{3,6\}$, for $\mathbf{S}(3) = \mathbf{S}(6) = 3$ and $3! = 1 \cdot 2 \cdot 3$; so $3! \in S_-(3)$; $S_-(4) = \{4,8,12,24\}$, for $\mathbf{S}(4) = \mathbf{S}(8) = \mathbf{S}(12) = \mathbf{S}(24) = 4$ and $4! = 1 \cdot 2 \cdot 3 \cdot 4$; so $4! \in S_-(4)$; $S_-(5) = \{5,10,15,20,30,40,60,120\}$, for $\mathbf{S}(5) = \mathbf{S}(10) = \mathbf{S}(15) = \mathbf{S}(20) = \mathbf{S}(30) = \mathbf{S}(40) = \mathbf{S}(60) = \mathbf{S}(120) = 5$ and $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$; so $5! \in S_-(5)$.

$S_-(6) = \{9,16,18,36,45,48,72,80,90,144,180,240,360,720\}$; so $6! \in S_-(6)$.

*Proposition 1*. Let $a \in \mathbb{N}^*$. Then:
i) $a$ is a prime number, if and only if $a^a \in S_-(a^2)$ and $a \neq 1$. ii) If $a$ is prime, then $a \in S_-(a)$.

The demonstration results from proposition 2 of [6].
*Application* 1. Determine **S** (3125).
*Solution*. It is found that 3125=$5^5$, and 5 is prime. Then according to the *proposition* 1 $5^5 \in S_-(5^2)$, i.e. 3125$\in S_-(25)$ and **S** (3125) = 25.

*Lemma* 1.
i) If $a \in S_-(9)$, then $\bar{s}(a) = 9$. ii) If $\mathbf{S}(n) \geq 6$, then $\bar{s}(\mathbf{S}(n)!) = 9$. iii) $a! \in S_-(a)$ şi $S_-(a) \subset \mathbb{N}^*$, for any $a \in \mathbb{N}^*$. iv) If $a$ is prime, then $a \in S_-(a)$.
*Proposition* 2. If $\propto \geq 1, k \geq 2$ şi $a = {p_1}^{\propto} \cdot \ldots \cdot {p_k}^{\propto}$, is its canonical decomposition, then $a \in S_-(\propto \cdot p_k)$.
The justification results from consequence 1.3 of [6].
*Application* 2. Determine natural numbers $x$ and $y$ such that 10222234378671145$\in S_-(x)$ and 101578766438334998$\in S_-(y)$.

*Solution*. Because 10222234378671145=5·2044446875734229, according to the proposition 2 results *x*=2044446875734229∈Prim and S_(2044446875734229)∋10222234378671145.

Because 10157876643834998=2·13·659·592849109597, according to the proposition 2 results *y*=592849109597∈Prim and S_(592849109597)∋10157876643834998.

*Application* 3. Let *a* = 100895598169, the number that P. Fermat, in the 17th century according to [1], decomposed into prime factors as follows: *a*=112303·898423. Then S_(898423)∋*a*, based on proposition 2.

## 3. A representation of $\mathbb{N}^*$

Let be the relation on$\mathbb{N}^*$, denoted by ~ (tilde) and defined as follows:

$$\forall x,y \in \mathbb{N}^*,\ x \sim y \Leftrightarrow \mathbf{S}(x) = \mathbf{S}(y).$$

The relation ~ is an equivalence relation (known result). Since $\mathbf{S}(n!) = n$, the equivalence class of $n \in \mathbb{N}^*$, is denoted by $[n]$ or $\hat{n}$, where $\hat{n} = \{x \in \mathbb{N}^* | x \sim n!\}$, i.e. $\hat{n} = \{x \in \mathbb{N}^* | \mathbf{S}(x) = n\}$=$S_-(n)$.
Then the quotient set $\mathbb{N}^*/\sim$ is made up of the equivalence classes of $\mathbb{N}^*$ in relation to the equivalence relation ~. That is, each equivalence class is the preimage of a number *n* by the function S_ and *n*! is a representative of the equivalence class and the notation $\widehat{n!}$ can be used in addition to the notation $\hat{n}$.

*Theorem* 1. The following representation is true:

$$\mathbb{N}^* = \bigcup_{n \in \mathbb{N}^*} \widehat{n!}.$$

In addition, $\{n! | n \in \mathbb{N}^*\}$ is a system of equivalence classes representatives, and $\{\hat{n} | n \in \mathbb{N}^*\}$ forms an infinitely countable partition of $\mathbb{N}^*$.
This results from the properties that any quotient set has.

*Theorem* 2. Let $n \in \mathbb{N}^*$.
i) If the equivalence class $\hat{n}$, where *n* is considered representative of the class, contains a prime number *m*, then *m* is the greatest common divisor of all elements in the class, being also the smallest representative of the class in relation to the relation ≤ . In this case, *m* is the only prime number in the class.

The reciprocal of the statement is not generally true.
ii) The equivalence class $\hat{n}$ does not contain any prime number, if the greatest common divisor of all the elements in the class is not a class representative. In this case, the class does not contain any prime numbers.
iii) A system of representatives of equivalence classes is given by the infinitely countable set:

$$\{1, 2, 6, 24, 120, 720, 5040, 40320, 362880, \dots, n!, \dots\}.$$

*Demonstration*.
i) We admit that in class $\hat{n}$ there is a prime number $m$. Then $m \sim n!$, which implies $\mathbf{S}(m) = \mathbf{S}(n!) = m$, where $m \in \hat{m} \cap \hat{n}$, i.e. $\hat{m} = \hat{n}$ and $m$ is representative of the class $\hat{n} = S_{-}(m)$, and $\mathbf{S}(n) = m$. Then $n = k \cdot m$, $k \in \mathbb{N}^*$, such that $\mathbf{S}(k) \leq m$, otherwise $\mathbf{S}(n) \neq m$. So $m|n$, i.e. $m$ divides any representative of the class $\hat{n}$. Let $d$ the greatest common divisor of all elements in the class $\hat{n}$. Then $d|m$ and $m|d$, because $m$ is a common divisor of the elements of class $\hat{n}$, and therefore $m=d$. Since $d$ is unique, $m$ is the only prime number in the class. Since any representative of the class is a multiple of $m$, $m$ is the smallest representative of the class $\hat{n}$ in relation to the relation $\leq$.
The reciprocal of the statement is not true in general, because 4 is the greatest common divisor of all the elements of class $\hat{4} = S_{-}(4)$, being also a representative of the class because $\mathbf{S}$ (4) = 4, in the same time not being prime.
ii) We assume that the greatest common divisor of all elements in class $\hat{n}$ is not a representative of the class. By reduction to the absurd, we assume that there is a prime number $m$ in this class. Then according to i) $m$ coincides with the greatest divisor of the elements of the class, being also the representative of the class, contradiction! It follows that in class $\hat{n}$ there is no prime number, i.e. any element of the class is a compound number.
iii) It results from the fact that $n! \in \hat{n}$, for any $n \in \mathbb{N}^*$.
Returning to application 3, we can write $\hat{a} = \widehat{898423} = \widehat{898423!}$, according to the above.

## 4. Results regarding equivalence classes and $\mathbf{S}_{-}$

*Proposition* 3. Let $n \in \mathbb{N}^*$. The following statements are true:
a) if $n$ is a prime number, then the canonical decomposition of $n!$ is:

$$n! = 2^{g(n,2)} \cdot 3^{g(n,3)} \cdot \dots \cdot n,$$

where $g(n,2) \geq g(n,3) \geq \cdots \geq 1$ and $\mathbf{S}\left(p^{g(n,p)}\right) < n$, for any $p$ prime and $p<n$.

b) if $n$ is a compound number, then the canonical decomposition of $n!$ is:

$$n! = 2^{g(n,2)} \cdot 3^{g(n,3)} \cdot \dots \cdot k^{g(n,k)},$$

where $k$ is the greatest prime number less than $n$, $g(n,2) \geq g(n,3) \geq \cdots \geq g(n,k)$ and there is at least one prime $p$, $p<n$, so that $\mathbf{S}\left(p^{g(n,p)}\right) = n$.
*Demonstration*.
a) Equality is obtained based on the significance of the function g, and the series of inequalities is obvious. Because $\mathbf{S}(n!) = \max\left\{\mathbf{S}\left(2^{g(n,2)}\right), \mathbf{S}\left(3^{g(n,3)}\right), \dots, \mathbf{S}(n)\right\} = n$, $n$ being prime, it results $\mathbf{S}\left(p^{g(n,p)}\right) \leq n$, for any $p$ prime and $p<n$, and $\mathbf{S}(n) = n$, of which:

$$p^{g(n,p)}|\mathbf{S}(p^{g(n,p)})! \text{ and } p^{g(n,p)}|n!.$$

Because *n* is prime and *n* does not contribute any *p* to the result of counting *g*(*n*, *p*) in *n*!, we have $p^{g(n,p)}|(n-1)!$. Then $\mathbf{S}(p^{g(n,p)}) \leq n-1 < n$.

b) Equality is obtained based on the significance of the function g, and the series of inequalities is obvious. Because $\mathbf{S}(n!) = \max\{\mathbf{S}(2^{g(n,2)}), \mathbf{S}(2^{g(n,2)}), \dots, \mathbf{S}(k^{g(n,k)})\} = n$, there is at least a prime *p*, *p* < *n* for which $\mathbf{S}(p^{g(n,p)}) = n$.

*Exemple*.

1) $6! = 2^{g(6,2)} \cdot 3^{g(6,3)} \cdot 5^{g(6,5)} = 2^4 \cdot 3^2 \cdot 5$ , and $\mathbf{S}(2^4) = \mathbf{S}(3^2) = 6, \mathbf{S}(5) = 5$.

2) $7! = 2^{g(7,2)} \cdot 3^{g(7,3)} \cdot 5^{g(7,5)} \cdot 7 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$, and $\mathbf{S}(2^4) = \mathbf{S}(3^2) = 6, \mathbf{S}(5) = 5$, **S(7)=7**.

*Consequence* 3.1. Under the conditions of proposition 3a, $S_-(n)$ contains a single number *a*, called **the maximal factor corresponding to** *n* !, with the following two properties:

1) *a* is a factor of the canonical decomposition of *n* !; 2) S (*a*) = *n*. In fact, the maximal factor is *n*.

*Consequence* 3.2. Under the conditions of proposition 3a, $a \in S_-(n)$ if and only if $a = n \cdot q$, $q \in \mathbb{N}^*$ and $\mathbf{S}(q) \leq n$.

*Demonstration*.

**Necessity**: we assume $a \in S_-(n)$. Then $\mathbf{S}(a) = n$. Assuming that *a* has the canonical decomposition $a = p_1^{\alpha_1} \cdot \ldots \cdot p_m^{\alpha_m}$, $\mathbf{S}(a) = \max\{\mathbf{S}(p_j^{\alpha_j}) | j = \overline{1,m}\} = n$, that which implies the existence of *i*, 1≤*i*≤*m*, with $\mathbf{S}(p_i^{\alpha_i}) = \mathbf{S}(n) = n$. Results $p_i = n$ and $\alpha_i = 1$, and on the other hand $a = n \cdot q$, where $q = \prod_{j=1, j\neq i}^{m} p_j^{\alpha_j}$.

Since **S** (*a*) = max {**S** (*n*), **S** (*q*)} = *n* and **S** (q) = $\max\{\mathbf{S}(p_j^{\alpha_j}) \leq n | j = \overline{1,m}, j \neq i\}$, results $\mathbf{S}(q) \leq n$.

**Sufficiency**: we assume $a = n \cdot q$, $q \in \mathbb{N}^*$ and $\mathbf{S}(q) \leq n$. Then $\mathbf{S}(a) = \max\{\mathbf{S}(n), \mathbf{S}(q)\} = n$, because $\mathbf{S}(n) = n$. So $a \in S_-(n)$.

*Consequence* 3.3. Under the conditions of proposition 3a, let $a \in S_-(n)$ with $a = n \cdot q$, $q \in \mathbb{N}^*$ and it is assumed that *a* has the canonical decomposition $a = p_1^{\alpha_1} \cdot \ldots \cdot p_m^{\alpha_m}$ from the demonstration of the consequence 3.2. Then:

a) $p_i \in \{2,3,\dots,n\} \cap$ Prim and $\alpha_i \leq g(n, p_i)$, for any 1≤*i*≤*m*.

*b*) $q = \prod_{j=1, j\neq i}^{m} p_j^{\alpha_j}$, $p_i^{\alpha_i} = n$ and *a*=*n*·*q*.

c) $\widehat{n}! = S_-(n) = \{a \in \mathbb{N}^* | a = n \cdot q, q < n!, q \in \mathbb{N}^*\}$.

*Lemma* 3. Let *p*∈Prim, $\alpha \in \mathbb{N}^*$ and *A* be the set of solutions of the inequality $g(x,p) \geq \alpha$, with *x* unknown, and *n* = min*A*. Then:

1) $p^{\alpha} \in S_-(n)$;

2) if $g(n-1,p) = \alpha - k$, *k*≥2, then $p^{\alpha-k+1}, \dots, p^{\alpha} \in S_-(n)$;

3) under the conditions from 2), if $a \in \mathbb{N}^*$ with $\mathbf{S}(a) \leq n$ and *a* is prime with *p*, then $a \cdot p^{\alpha-k+1}, \dots, a \cdot p^{\alpha} \in S_-(n)$.

*Proposition* 4. If $n$ is a compound number whose canonical decomposition has $m$ factors, $1\leq m<n$, so that $n = p_1^{\propto_1} \cdot \ ... \cdot p_m^{\propto_m}$, then:

a) the canonical decomposition of $n!$ it also contains $m$ factors, called **maximal factors corresponding to** $n$ !, of the form $p_1^{g(n,p_1)}, \dots, p_m^{g(n,p_m)}$, having the property $\mathbf{S}\left(p_i^{g(n,p_i)}\right) = n$, for any $i=\overline{1,m}$;

b) any other factor $p^{\propto}$ in the canonical decomposition of $n$ !, with $p\neq p_i$ and $i=\overline{1,m}$, has $\mathbf{S}(p^{\propto}) < n$.

c) if $p^{g(n,p)}$ is a maximal factor, so that there exists $k\in\mathbb{N}^* - \{1\}$, with $g(n-1,p) = g(n,p) - k$, then $\mathbf{S}\left(p^{g(n,p)-k+j}\right) = n$, for any $j=\overline{1,k}$. In this case, the factors $p^{g(n,p)-k+1}$, ..., $p^{g(n,p)-1}$ are called **quasimaximal factors corresponding to** $p^{g(n,p)}$ and belong to $S_-(n)$. If there is, $k$ corresponds to $p$.

d) $a\in S_-(n)$ if and only if $a=f\cdot q\leq n!$, where f is a maximal factor, a quasi-maximal factor, a finite product of distinct maximal or quasi-maximal factors and which do not have the same basis, and $q$ is 1 or a product of factors, from the canonical decomposition of $n$ !, which are neither maximal nor quasi-maximal.

*Demonstration*.

a) Let $n! = q_1^{\beta_1} \cdot q_2^{\beta_2} \cdot ... \cdot q_k^{\beta_k}$, the canonical decomposition of n !. Then there are indices $1 \leq i_1 < i_2 < \cdots < i_m \leq k$, so that $q_{i_j} = p_j$, $\forall j=\overline{1,m}$. Since $\beta_{i_j} = g(n,p_j)$, it results: $\propto_j\leq g(n,p_j) = \beta_{i_j}$, $\forall j=\overline{1,m}$.

Since $(n-1, n) = 1$, $n-1$ does not contribute any $p_j$ to the count, while $n$ contributes $\propto_j$ to the result of counting $g(n,p_j)$. Then $p_j^{g(n,p_j)} \nmid (n-1)!$ but $p_j^{g(n,p_j)}|n!$ and so $\mathbf{S}\left(p_j^{g(n,p_j)}\right) = n$, $\forall j=\overline{1,m}$.

b) Let $p^{\propto}$ be a factor in the canonical decomposition of $n$ !, with $p\neq p_i$ and $i=\overline{1,m}$. Then there are $1\leq j\leq k$, $j\neq i$ and $i=\overline{1,m}$, so that $p^{\propto} = q_j^{\beta_j}$, $p = q_j$ and $\propto = \beta_j = g(n,q_j)$. Since $q_j \neq p_i$, $\forall\ i=\overline{1,m}$, $n$ does not contribute with any $q_j \neq p_i$, $\forall\ i=\overline{1,m}$ to the result of counting $g(n,q_j)$, instead there exists $s\in\mathbb{N}^*$ so that:

$$\beta_j = g(n,q_j) = g(n-1,q_j) = \cdots = g(n-s,q_j) > g(n-s-1,q_j).$$

Therefore $q_j^{\beta_j}|(n-s)!$ and $q_j^{\beta_j} \nmid (n-s-1)!$, of where it is obtained:

$$\mathbf{S}(p^{\propto}) = \mathbf{S}\left(q_j^{\beta_j}\right) = n - s < n.$$

c) It results from the lemma 3.

d) **Necessity**: either $a\in S_-(n)$. Then $a|n!$ and $a\leq n$ !. Let $a=t_1^{\delta_1} \cdot ... \cdot t_r^{\delta_r}$ the canonical decomposition of $a$. Since $\mathbf{S}(a) = \max\left\{\mathbf{S}\left(t_h^{\delta_h}\right) | h = \overline{1,r}\right\} = n$, there is at least one index $i$, $1\leq i\leq r$, and corresponding to it there is a single index $j_i$, $1\leq j_i \leq m$, so that $p_{j_i}^{g\left(n,p_{j_i}\right)}$ is a maximal factor of $n$ !, where: or $t_i^{\delta_i} = p_{j_i}^{g\left(n,p_{j_i}\right)}$ which implies $t_i = p_{j_i}$ and $\delta_i = g\left(n,p_{j_i}\right)$ or $t_i^{\delta_i}$ is a quasi-maximal factor corresponding to $p_{j_i}^{g\left(n,p_{j_i}\right)}$.

Let $f$ be the product of all distinct maximal or quasi-maximal factors that do not have the same basis, which make up the canonical decomposition of $a$. Obviously, $f|a$ and $f\leq a$.

If $f<a$, let be $q$ the product of all the distinct factors that make up the canonical decomposition of $a$, which are neither maximal nor quasi-maximal. Obviously, $q|a$ and $(f,q) = 1$.

As any factor in the canonical decomposition of $a$ is either maximal, or quasi-maximal, or is neither maximal nor quas-imaximal one can write $a=f\cdot q$.

If $f=a$ we obtain $q = 1$, i.e. there are no factors in the canonical decomposition of $a$, which are neither maximal nor quasi-maximal.

**Sufficiency**: we assume $a=f\cdot q\leq n!$, where $f$ is a maximal factor, a quasi-maximal factor, a finite product of distinct maximal or quasi-maximal factors and which do not have the same basis, and $q$ is 1 or a product of distinct factors, from the canonical decomposition of $n!$, which are neither maximal nor quasi-maximal.

Then $\mathbf{S}(a) = \max\{\mathbf{S}(f), \mathbf{S}(q)\} = n$, because $(f, q) = 1$, $\mathbf{S}(f) = n$ and $\mathbf{S}(q) < n$. So $a\in \mathrm{S}_{-}(n)$.

According to proposition 3c) it makes sense, for $n$ compound number, the smallest qusimaximal factor corresponding to a fixed maximal factor, called **the minimum quasi-maximal factor**, as the minimum of the quasi-maximal factors corresponding to it, when they exist. Also, if any, any quasi-maximal factor corresponding to a maximal factor is a number between the minimum quasi-maximal factor and the maximal factor.

It is also found that $f$ cannot be the product between a maximal factor and a quasi-maximal factor corresponding to it, because otherwise $\mathbf{S}(f) > n$, because the exponent of the base of the maximal factor increases.

*Consequence* 4.1. Under the conditions of sentence 4, the equivalence class $\hat{n}$ is the set:

$$\widehat{n}! = \{a \in \mathbb{N}^* | a = f \cdot q \leq n!\} = \mathrm{S}_{-}(n) = \hat{n}.$$

In addition, the existence of quasi-maximal factors corresponding to a maximal factor only makes sense if $n$ is compound.

*Consequence* 4.2. i) If $n$ is prime, then $\min\widehat{n}! = \min\mathrm{S}_{-}(n) = n$. In this case, $n$ is the class representative. ii) Any be $n\in\mathbb{N}^*$, $\max\widehat{n}! = \max\mathrm{S}_{-}(n) = n!$ and the set $\mathrm{S}_{-}(n)$ is finite with $\mathrm{card}\mathrm{S}_{-}(n) \leq n!$.

**Conclusion**

The relevance of the developed mathematical sentences gives the premises for future, complex research.

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