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AN ALGORITHM FOR CALCULATING SMARANDACHE'S FUNCTION AND WICH USING LEGENDRE'S FORMULA

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Abstract: *The paper presents a calculation algorithm for the values of the function* ***S****, defined by Fl. Smarandache [6], [1], [4], [5], an algorithm that uses the writing of numbers in base ten and it is based on Legendre's formula and some theoretical results. It differs from the one presented in [8], which avoids factorization, and from the one presented in [6], which requires writing on a generalized basis. Then a characterization of a prime number is given. Finally, a numerical application is presented.*

Key Words: *divisibility of numbers, writing a number in a base, generalized base, the integer part of a number, prime number, compound number, partial function, algorithm.*

JEL Classification: *C00, C02.*

1. Introduction

The function of Fl. Smarandache can be defined as follows [4], [5], [6]:
S:N^*→N^*, S(n)=min{x∈N^* ├|n├|x!┤┤}.

The function S is well defined because n|n!┤, ∀n∈N^*, that is, the set {x∈N^* ├|n├|x!┤┤} is non-empty and being included in ℕ, it is ordered ascending, its minimum value exists, it belongs to the set and it is unique. In general, the Smarandache function is not injective, because S (27) = S (81) = 9 although 27 ≠ 81.

2. Another characterization of prime numbers using the S function

For $a\in\mathbb{N}^*$ it is called canonical decomposition or the factorization of *a*, the representation

$a=p_1^{\propto_1}\cdot p_2^{\propto_2}\cdot\ldots\cdot p_k^{\propto_k}$, where $k\in\mathbb{N}^*$, $\propto_1,\ldots,\propto_k\in\mathbb{N}$ and $1<p_1<\cdots<p_i<\cdots<p_k$ are prime numbers [3].

Proposition 1. Let there be $\min\{\propto_1,\ldots,\propto_k\}=\propto$ and $M=\max\{p_1\cdot\propto_1,\ldots,p_k\cdot\propto_k\}$. Then the following inequalities take place:

a) If k=1, then $a=p^{\propto}$, p prime number and $\propto\in \mathbb{N}^*$, **S**(a)≤M=$\propto\cdot p$. In addition, if $p^2 < \text{M}$, then **S**(a)<M.
b) If k≥2, then: $\propto\cdot p_k \le \mathbf{S}(a) \le \text{M}$.

Proof.

a) For k=1, $a=p^{\propto}$ and M=$\propto\cdot p$ take place. Because $p^{\propto}|$M!, results **S**(a)≤M. Three cases are possible:

a1) 1≤$\propto< p$, then p<2p<...<$\propto\cdot p$=M and the number of p starting with p and ending with $\propto\cdot p$ is exactly $\propto$. Then $p^{\propto}|(\propto\cdot p)! = \text{M}!$ and $p^{\propto} \nmid (\propto\cdot p-1)! = (\text{M}-1)!$. Therefore **S**($a$)=M.

a2) $\propto = p$, then $a=p^p$, $p < 2p < \cdots < p^2-1 < p^2 = \text{M}$, and the number of p starting with p and ending with p^2 is p + 1, because p^2contributes two of p to the counting, and the number of p that begins with p and ends with p^2-1 is p -1. Results $p^p|$M! and $p^p \nmid (\text{M}-1)!$. Then **S**(p^p)=M=p^2.

a3) $\propto > p$, then $p< 2p < \cdots < p^2 < p^2+1 < \cdots <\propto\cdot p = \text{M}$, the number of p starting with p and ending with $\propto\cdot p$ is at least p + 1, because p^2 contributes two of p to the counting. Is obtained $p^{\propto}|(\propto\cdot p-1)!$ and **S**(a)≤ $(\propto\cdot p-1) < \text{M}$.

b) For k≥2, because $p_i^{\propto_i}|$M! any i results $a|$M! and then **S**(a)≤M.

It is observed that $p_k! = 1\cdot 2\cdot \ldots\cdot p_1\cdot\ldots\cdot p_i \ldots\cdot p_k$ and

$$(\propto\cdot p_k)! = 1\cdot 2\cdot\ldots\cdot p_1\cdot\ldots\cdot p_i\cdot\ldots\cdot p_k\cdot(p_k+1)\cdot\ldots\cdot(2p_k)\cdot(2p_k+1)\cdot\ldots\cdot(\propto p_k-1)\cdot(\propto p_k).$$

Because $p_i < p_k$ and $\propto p_i <\propto p_k$ any $i=\overline{1,k-1}$, results: $p_i^{\propto}|(\propto p_k)!$, $\forall i=\overline{1,k}$ and $p_1^{\propto}\cdot\ldots\cdot p_i^{\propto}\cdot\ldots\cdot p_k^{\propto}|a|\mathbf{S}(a)!$wherefrom $p_1^{\propto}\cdot\ldots\cdot p_i^{\propto}\cdot\ldots\cdot p_k^{\propto}|\mathbf{S}(a)!$ and $p_1^{\propto}\cdot\ldots\cdot p_i^{\propto}\cdot\ldots\cdot p_k^{\propto} \le a$.

If $p_1^{\propto}\cdot\ldots\cdot p_i^{\propto}\cdot\ldots\cdot p_k^{\propto} = a$, then **S**($a$)= $\propto p_k \le \text{M}$, because

$$p_1 < \cdots <\propto p_1 < \ldots <\propto p_2 < \cdots <\propto p_k,$$

and in the factorial $(\propto p_k)!$ the number of p_k starting with p_k and ending with $\propto p_k$ is exactly $\propto$, which involves $a|(\propto p_k)!$ and $a\nmid(\propto p_k-1)!$. Then $\propto p_k = \mathbf{S}(a) \le \text{M}$.

If $p_1^{\propto}\cdot\ldots\cdot p_i^{\propto}\cdot\ldots\cdot p_k^{\propto} < a$, then $\propto p_k < \mathbf{S}(a)$≤M because $a\nmid(\propto p_k)!$.

Example 1. Let a=3^5. Then $\propto\cdot p = \text{M} = 3\cdot 5 = 15$. Because $3^2 < 15$, case a3) of the demonstration is applied and it is found that **S**(a)=12<15, because:

$$3^5|12! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot(3\cdot 2)\cdot 7\cdot 8\cdot(3\cdot 3)\cdot 10\cdot 11\cdot(3\cdot 4) \text{ and } 3^5 \nmid 11!.$$

Exemple 2. Let a=$3^3\cdot 5^3\cdot 23^3$. Then $\propto$=min{3,3,3} = 3, M=max{9,15,69} = 69, $p_k = 23$ and case b) is applied, which involves **S**(a)=$\propto\cdot p_k = 69 = \text{M}$.

Consequence 1.1. $\mathbf{S}(a) =\propto\cdot p$, if a has a canonical decomposition $a=p^{\propto}$, where 1≤$\propto\le p$.

Consequence 1.2. $\mathbf{S}(a) < \propto \cdot p$, if a has a canonical decomposition $a = p^{\propto}$, where $p < \propto$.

Consequence 1.3. $\mathbf{S}(a) = \propto \cdot p_k$, if a has a canonical decomposition $a = p_1{}^{\propto} \cdot \ldots \cdot p_i{}^{\propto} \cdot \ldots \cdot p_k{}^{\propto}$, where $k \geq 2$ and $\propto \geq 1$.

The following statement is a characterization of a prime number.

Proposition 2. Let $a \neq 1$. $\mathbf{S}(a^a) = a^2$ if and only if a is a prime natural number.

Proof. The sufficiency: let $\mathbf{S}(a^a) = a^2$ and we assume by reduction to the absurd that $a \neq 1$ is not a prime number. Then a has the canonical decomposition $a = p_1{}^{\propto_1} \cdot p_2{}^{\propto_2} \cdot \ldots \cdot p_k{}^{\propto_k}$, where $\propto_1, \ldots, \propto_k \in \mathbb{N}^*$ and $1 < p_1 < \cdots < p_i < \cdots < p_k$ are prime numbers, such that $k \geq 2$ or $k = 1$ and $\propto_1 \geq 2$. Thus: $a^a = p_1{}^{a \cdot \propto_1} \cdot \ldots \cdot p_k{}^{a \cdot \propto_k}$ and $\max\{a \cdot \propto_1 \cdot p_1, \ldots, a \cdot \propto_k \cdot p_k\} = a\mathrm{M}$, where $\mathrm{M} = \max\{\propto_1 \cdot p_1, \ldots, \propto_k \cdot p_k\}$.

Then, according to proposition 1 it must be $\mathbf{S}(a^a) \leq a \cdot \mathrm{M}$, meaning $a^2 \leq a\mathrm{M}$ or $a \leq \mathrm{M}$ and

$$p_1{}^{\propto_1} \cdot p_2{}^{\propto_2} \cdot \ldots \cdot p_k{}^{\propto_k} \leq \max\{\propto_1 \cdot p_1, \ldots, \propto_k \cdot p_k\}. \qquad (*)$$

The following situations are possible:

c1. $k \geq 2$. In this case, there is i, $1 \leq i \leq k$, and $\mathrm{M} = \propto_i \cdot p_i$. Then the relation (*) becomes

$$p_1{}^{\propto_1} \cdot p_2{}^{\propto_2} \cdot \ldots \cdot p_k{}^{\propto_k} \leq \propto_i \cdot p_i. \qquad (**)$$

It is observed that $p_i{}^{\propto_i} \geq \propto_i \cdot p_i$, and $p_j{}^{\propto_j} > 1$, any $j \neq i$, which involves $p_1{}^{\propto_1} \cdot p_2{}^{\propto_2} \cdot \ldots \cdot\cdot\, p_k{}^{\propto_k} > p_i{}^{\propto_i}$ and so the relation (**) is false. Then the number a is prime.

c2. $k = 1$ and $\propto_1 \geq 2$. Then $a = p_1{}^{\propto_1}$ and the relationship (*) becomes $p_1{}^{\propto_1} \leq \propto_1 \cdot p_1$, strict inequality being false. All that remains is the equality situation, this could be true for $\propto_1 = 1$, which would contradict the hypothesis, or for $\propto_1 = 2$ and $p_1 = 2$, which leads to the following calculation: $a = 4, a^a = 4^4 = 256$ și $\mathbf{S}(4^4) = 10 \neq 16 = 4^2$, contradiction! because $10! = 3628800 = 256 \cdot 14175$ and $256 \nmid 9!$. Then the number a is prime.

The necessity: we assume that a is a prime number. Then $\mathbf{S}(a^a) = a^2$, according to case a2) from the demonstration of proposition 1.

Comment. The statement "if a is a prime number, then $\mathbf{S}(a^a) = a^2$" was proved, in another way, in [7].

A characterization of compound numbers is established by the following statement, deduced from proposition 2 and the consequence 1.2.

Consequence 2.1. $a\neq 1$ is a natural compound number, if and only if **S**(a^a)<a^2.

3. A connection between the S function and Legendre's formula

In the above, it is not specified how to calculate **S**(a), in the case where canonical decomposition of the natural number a contains at least two different nonzero exponents, different from each other.

Legendre's formula is needed to treat this case [2], [3], [6], [7]:

$$exp_{n!}(p) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots,$$

Where x in the parentheses $[x]$ denotes the integer part of x, p is a natural prime number, $n\in\mathbb{N}^*$ and $exp_{n!}(p)$ means the exponent of the greatest power at which p appears in the decomposition of $n!$; in other words, $exp_{n!}(p)$ is the maximum number of occurrences of p in the writing of $n!$.

For example, let n=91 and p=7. Then the highest power at which 7 appears in the decomposition of 91! is:

$$exp_{91!}(7) = \left[\frac{91}{7}\right] + \left[\frac{91}{7^2}\right] + \left[\frac{91}{7^3}\right]=13+1+0=14.$$

It can be noted that:

$$91!=1\cdot\ldots\cdot 7\cdot\ldots\cdot(2\cdot 7)\cdot\ldots\cdot(3\cdot 7)\cdot\ldots\cdot(4\cdot 7)\cdot\ldots\cdot(5\cdot 7)\cdot\ldots\cdot(6\cdot 7)\cdot\ldots\cdot(7\cdot 7)\cdot\ldots$$

$$\cdot(8\cdot 7)\cdot\ldots\cdot(9\cdot 7)\cdot\ldots\cdot(10\cdot 7)\cdot\ldots\cdot(11\cdot 7)\cdot\ldots\cdot(12\cdot 7)\cdot\ldots\cdot 90\cdot(13\cdot 7),$$

that is, $7^{14}|91!$, in other words, 7 appears 14 times in the writing of 91!. Moreover, **S**(7^{14})=91, because $7^{14}\nmid 90!$.

Returning to Legendre's formula, we find that for any $n\in\mathbb{N}^*$ and any p prime natural number, the sequence with positive integers $\left(\left[\frac{n}{p^k}\right]\right)_{k\in\mathbb{N}^*}$ is stationary, the natural exponent existing $k_{p,n}$ such that $p^{k_{p,n}}\leq n<p^{k_{p,n}+1}$ and $\left[\frac{n}{p^k}\right] = 0, \forall k \geq k_{p,n} + 1$. Moreover, $\left[\frac{n}{p^{k_{p,n}}}\right] \neq 0$ and $\lim_{k\to\infty}\left[\frac{n}{p^k}\right] = 0$, resulting that the sum $exp_{n!}(p)$ contain $k_{p,n}$ non-zero terms and $exp_{n!}(p) \in \mathbb{N}$.

Remark 1. The following inequalities take place:

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$$\left[\frac{n}{p}\right] > \left[\frac{n}{p^2}\right] > \cdots > \left[\frac{n}{p^{k_{p,n}+1}}\right] = 0.$$

To establish a connection between the **S** function and the Legendre formula, the following function is defined:

$$g{:}\mathbb{N}^* \times \text{Prim} \to \mathbb{N}, \quad g(n,p) = \begin{cases} exp_{n!}(p), & n \geq p \\ 0, & 1 \leq n < p \end{cases},$$

where Prim is the set of natural prime numbers.

Remark 2. There are $n_1 \neq n_2$ and $p \in$Prim, such that $g(n_1, p) = g(n_2, p)$.

Indeed, let $n_1 = 94$, $n_2 = 95$ and p=2. Then $k_{2,94} = k_{2,95} = 6$ and:

$$g(94,2) = \left[\frac{94}{2}\right] + \left[\frac{94}{4}\right] + \left[\frac{94}{8}\right] + \left[\frac{94}{16}\right] + \left[\frac{94}{32}\right] + \left[\frac{94}{64}\right] + \left[\frac{94}{128}\right] = 47 + 23 + 11 + 5 + 2 + 1 = 89;$$

$$g(95,2) = \left[\frac{95}{2}\right] + \left[\frac{95}{4}\right] + \left[\frac{95}{8}\right] + \left[\frac{95}{16}\right] + \left[\frac{95}{32}\right] + \left[\frac{95}{64}\right] + \left[\frac{95}{128}\right] = 47 + 23 + 11 + 5 + 2 + 1 = 89,$$

because $\forall k \geq 7$ take place: $\left[\frac{94}{2^k}\right] = \left[\frac{95}{2^k}\right] = 0$.

Remark 3. There are $\propto \in \mathbb{N}^*$ and $p \in$Prim fixed so that the equation $g(x,p) = \propto$, with $x \in \mathbb{N}^*$ unknown, it has no solution in $\mathbb{N}^*$.

Indeed, either $\propto$=90 and p=2. Then: $g(95,2) = 89 < 90 < 94 = g(96,2)$.

As there is no x natural between 95 and 96 the equation $g(x, 2) = 90$ has no solution in $\mathbb{N}^*$.

Proposition 3. Let n, $p \in \mathbb{N}^*$, p prime. Then: a) $k_{p,n} = [\log_p n]$ or $k_{p,n} = \left[\frac{\ln n}{\ln p}\right]$; *b)* $g(n,p) = \sum_{i=1}^{T} \left[\frac{n}{p^i}\right]$, $T = k_{p,n}$.

Example 3. $k_{2,95} = [\ln 95/\ln 2] = [6{,}5699] = 6$.

Proposition 4. Let $(n,\ p) \in \mathbb{N}^* \times$ Prim. Then $\mathbf{S}\left(p^{g(n,p)}\right) \leq n$, and equality is achieved when $p^{g(n,p)} \nmid (n-1)!$.

Proof. Because $g(n,p) \in \mathbb{N}$ makes sense $p^{g(n,p)} \in \mathbb{N}^*$ and there is $\mathbf{S}\left(p^{g(n,p)}\right)$. It follows from the definition of g that $p^{g(n,p)}|n!$, that is $\mathbf{S}\left(p^{g(n,p)}\right) \leq n$. If $p^{g(n,p)} \nmid (n-1)!$ is obtained $\mathbf{S}\left(p^{g(n,p)}\right) = n$.

Proposition 5. Let $\propto \in \mathbb{N}^*$ such that $\left[\frac{n}{p^{\propto}}\right] \neq 0$ and $\left[\frac{n}{p^{\propto+1}}\right] = 0$, then $k_{p,n} = \propto$ and the reciprocal.

Proof. Direct implication: we assume that $\propto \in \mathbb{N}^*$ such that $\left[\frac{n}{p^{\propto}}\right] \neq 0$ and $\left[\frac{n}{p^{\propto+1}}\right] = 0$. Then $p^{\propto} \leq n < p^{\propto+1}$, otherwise $\left[\frac{n}{p^{\propto+1}}\right] \neq 0$ and contradiction! Results $\left[\frac{n}{p^{\propto+1}}\right] = \left[\frac{n}{p^{k}}\right] = 0$, for any $k \geq \propto +1$, that is $k_{p,n} = \propto$.

The inverse implication results from the definition of $k_{p,n}$.

Proposition 6. Let $\propto \in \mathbb{N}^*$ and $p \in$Prim. In these conditions $k_{p,n} = \propto$, if and only if $n \in \left\{p^{\propto} + x \middle| x = \overline{0, p^{\propto+1} - p^{\propto} - 1}\right\}$.

Proof. Direct implication: we assume that $k_{p,n} = \propto \in \mathbb{N}^*$. Then $\left[\frac{n}{p^{\propto}}\right] \neq 0$, where from $p^{\propto} \leq n < p^{\propto+1}$ and $\left[\frac{n}{p^{\propto+j}}\right] = 0$ for any $j \geq 1$. Because $p^{\propto} \leq p^{\propto} + x < p^{\propto+1}$ results $\left[\frac{p^{\propto}+x}{p^{\propto}}\right] \geq \left[\frac{p^{\propto}}{p^{\propto}}\right] \neq 0$ for $\forall x = \overline{1, p^{\propto+1} - p^{\propto} - 1}$, but $\left[\frac{p^{\propto}}{p^{\propto+j}}\right] = \left[\frac{p^{\propto}+x}{p^{\propto+j}}\right] = 0$ for any $j \geq 1$ and for any $x = \overline{1, p^{\propto+1} - p^{\propto} - 1}$. Then $n \in \left\{p^{\propto} + x \middle| x = \overline{0, p^{\propto+1} - p^{\propto} - 1}\right\}$.

Inverse implication: we assume that $n \in \left\{p^{\propto} + x \middle| x = \overline{0, p^{\propto+1} - p^{\propto} - 1}\right\}$.

Then $\left[\frac{p^{\alpha}+x}{p^{i}}\right] \neq 0, \forall i = \overline{1, \propto}$ and $\forall x = \overline{0, p^{\propto+1} - p^{\propto} - 1}$, but $\left[\frac{p^{\alpha}+x}{p^{\propto+j}}\right] = 0$ for any $j \geq 1$ and for any $x = \overline{0, p^{\propto+1} - p^{\propto} - 1}$. Results $k_{p,n} = \propto$, for any $x = \overline{0, p^{\propto+1} - p^{\propto} - 1}$.

Proposition 7. It takes place $g(n,p) = g(n+1,p)$ if and only if $k_{p,n} = k_{p,n+1}$ and

$$\left[\frac{n}{p^k}\right] = \left[\frac{n+1}{p^k}\right], \forall k = \overline{1, k_{p,n}}.$$

Proof. We assume that $g(n,p) = g(n+1,p)$. Then $k_{p,n} \leq k_{p,n+1}$. We assume by absurdity that $k_{p,n} < k_{p,n+1}$. Results $k_{p,n+1} = k_{p,n} + r, r \geq 1$, where from $\left[\frac{n}{p^{k_{p,n+1}}}\right] = 0$ and $\left[\frac{n+1}{p^{k_{p,n+1}}}\right] \neq 0$, then how $\left[\frac{n}{p^i}\right] \leq \left[\frac{n+1}{p^i}\right]$, $\forall i = \overline{1, k_{p,n}}$ is obtained $g(n,p) < g(n+1,p)$ contradiction! Therefore $k_{p,n} = k_{p,n+1}$.

On the other hand, if there is the index j, $1 \leq j \leq k_{p,n}$, with $\left[\frac{n}{p^j}\right] < \left[\frac{n+1}{p^j}\right]$, then $g(n,p) < g(n+1,p)$ contradiction! Therefore $\left[\frac{n}{p^i}\right] = \left[\frac{n+1}{p^i}\right]$, $\forall i = \overline{1, k_{p,n}}$.

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The reverse implication results immediately.

4. A calculation algorithm for the function S

The algorithm presented in the paper, for determining the values of the function **S**, uses the writing of numbers in base ten and is based on the following theoretical results.

Proposition 8. The function g is increasing in the first argument when the second is fixed and decreasing in the second argument when the first is fixed.

Proof. Let $n_1, n_2 \in \mathbb{N}^*$, $n_1 < n_2$ and p∈Prim. Then $k_{p,n_1} \leq k_{p,n_2}$ and $\left[\frac{n_1}{p^i}\right] \leq \left[\frac{n_2}{p^i}\right]$, ∀$i$=$\overline{1, k_{p,n_1}}$. Therefore $g(n_1, p) \leq g(n_2, p)$. Let n∈$\mathbb{N}^*$, $p_1, p_2 \in$ Prim and $p_1 < p_2$. Then $k_{p_1,n} \geq k_{p_2,n}$ and $\left[\frac{n}{{p_1}^i}\right] \geq \left[\frac{n}{{p_2}^i}\right]$, for any i=$\overline{1, k_{p_2,n}}$. Results $g(n, p_1) \geq g(n, p_2)$.

Consequence 8.1. For ∝∈$\mathbb{N}^*$ and p∈Prim fixed, let inequation $g(x, p) \geq \propto$ with x∈$\mathbb{N}^*$ unknown. Then A, the set of solutions of the inequation, is nonempty.

Demonstration. Because the partial function $g(\cdot, p)$: $\mathbb{N}^* \rightarrow \mathbb{N}$ is monotonically increasing on a totally ordered set with values in a totally ordered set and $\lim_{x\to\infty} g(x, p) = \infty$, the set of solutions of the inequation is a nonempty subset of $\mathbb{N}$.

Consequence 8.2. If A is the set of solutions to the inequation $g(x, p) \geq \propto, \propto \in \mathbb{N}^*$ and n=min$A$, then $\mathbf{S}(p^{\propto}) = n$.

Demonstration. If x∈A, then $x! \vdots p^{g(x,p)} \vdots p^{\propto}$ and $p^{\propto} | x!$. Because A⊂$\mathbb{N}$ there is min$A$. Because n=min$A$ results n∈A, then $p^{\propto} | n!$ and $p^{\propto} \nmid (n-1)!$, because n=min$A$, therefore $\mathbf{S}(p^{\propto}) = n$.

Proposition 9. Let a∈$\mathbb{N}^*$ with canonical decomposition a=${p_1}^{\propto_1} \cdot ... \cdot {p_k}^{\propto_k}$, k∈$\mathbb{N}^*$. If was determined $\mathbf{S}({p_i}^{\propto_i})$ for any i=$\overline{1, k}$, then: $\mathbf{S}(a) = \max\{\mathbf{S}({p_i}^{\propto_i}) | i = \overline{1, k}\}$.

Demonstration. Let $\mathrm{S_M} = \max\{\mathbf{S}({p_i}^{\propto_i}) | i = \overline{1, k}\}$. Then there is an index j, 1≤j≤k, such that $\mathrm{S_M} = \mathbf{S}({p_j}^{\propto_j})$ and $\mathrm{S_M} \geq \mathbf{S}({p_i}^{\propto_i})$ for any i≠j. Because $\mathrm{S_M}! \geq \mathbf{S}({p_i}^{\propto_i})!$ any i≠j results ${p_i}^{\propto_i} | \mathrm{S_M}!$ any i, therefore $a | \mathrm{S_M}!$.

On the other hand, $a \nmid (S_M - 1)!$ because $S_M - 1 = \mathbf{S}(p_j^{\alpha_j}) - 1$ and which involves $p_j^{\alpha_j} \nmid (S_M - 1)!$. The relations $a|S_M!$ and $a \nmid (S_M - 1)!$ lead to $\mathbf{S}(a) = S_M$.

Comment. Proposition 9 is in line with the definition of the function **S** in [6].

According to consequence 8.2 and proposition 9, the algorithm for calculating the values of the function **S** for a natural number $a \in \mathbb{N}^*$, consists of the following steps:

I. Determine the canonical form of a. This has the form: $a = p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_k}$, $k \in \mathbb{N}^*$;

II. Calculate $\mathbf{S}(p_i^{\alpha_i})$ for each i, $1 \leq i \leq k$:

- if $1 \leq \alpha_i \leq p_i$ consequence 1.1 applies;
- if $\alpha_i > p_i$ go through the sub-stages:

II1. Determine A, the set of solutions of the inequation $g(x, p_i) \geq \alpha_i$;

II2. $\mathbf{S}(p_i^{\alpha_i}) = \min A$;

III. $\mathbf{S}(a) = \max\{\mathbf{S}(p_i^{\alpha_i}) \mid i = \overline{1,k}\}$.

For stage I there are special calculation algorithms. Sub-stage II1 is relatively simple.

5. Application

Let $a = 2^{90} \cdot 3^{27} \cdot 7^{12}$. Calculate **S** ($a$) by applying the above algorithm.

Step I. $a = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3}$, where $p_1 = 2, \alpha_1 = 90, p_2 = 3, \alpha_2 = 27, p_3 = 7$ și $\alpha_3 = 12$.

Step II.

i=1. Because $\alpha_1 > p_1$, apply sub-stage II1: consider the inequation $g(x, 2) \geq 90$; to determine A, the set of solutions of the inequation, it is observed that M=2·90=180>$2^2 = 4$, which involves $\mathbf{S}(2^{90}) < M$.

Given the monotony of the function g and $\left[\frac{86}{2}\right] + \left[\frac{86}{4}\right] = 43 + 21 = 64$ choose x= 86. $g(86,2) = 82$<90 is obtained, which involves increasing the value of x. Because $\left[\frac{88}{2}\right] + \left[\frac{88}{4}\right] = 44 + 22 = 66$, choose x=88.

$g(88,2) = 66 + 11 + 5 + 2 + 1 = 85<90$ is obtained, which involves increasing the value of *x*. Because $\left[\frac{92}{2}\right] + \left[\frac{92}{4}\right] = 46 + 23 = 69$, choose *x*=92. $g(92,2) = 69 + 11 + 5 + 2 + 1 = 88 < 90$ is obtained, which involves increasing the value of *x*. It is found that $g(94,2) = 47 + 23 + 11 + 5 + 2 + 1 = 89$, which involves increasing the value of *x*. For *x*=96, $g(96,2) = 48 + 24 + 12 + 6 + 3 + 1 = 94 > 90$ is obtained, which implies a decrease in the value of *x*. For *x*=95, $g(95,2) = 47 + 23 + 11 + 5 + 2 + 1 = 89$ is obtained.

Because $g(95,2)$ =89<90<94=$g(96,2)$, results *A*={96,97, ...}.

apply sub-stage II2. $\mathbf{S}(p_1{}^{\propto_1})$=$\mathbf{S}(2^{90}) = \min A = 96$.

i=2. Because $\propto_2 > p_2$, apply sub-stage II1: consider the inequation $g(x, 3) \geq 27$; to determine *A*, the set of solutions of the inequation, it is observed that M=3·27=81>$3^2 = 9$, which involves $\mathbf{S}(3^{27}) < M$. Noticing that $\left[\frac{81}{3}\right] - \left[\frac{81}{9}\right] = 18$, choose *x*=81-18=63. It is obtained:

$$g(63,3) = \left[\frac{63}{3}\right] + \left[\frac{63}{9}\right] + \left[\frac{63}{27}\right] + \left[\frac{63}{81}\right] = 21 + 7 + 2 + 0 = 30 > 27 =\propto_2.$$

This involves decreasing the value of *x* and we choose *x*=60. It is obtained:

$$g(60,3) = \left[\frac{60}{3}\right] + \left[\frac{60}{9}\right] + \left[\frac{60}{27}\right] + \left[\frac{60}{81}\right] = 20 + 6 + 2 + 0 = 28 > 27.$$

This involves decreasing the value of *x* and we choose *x*=57. It is obtained:

$$g(57,3) = \left[\frac{57}{3}\right] + \left[\frac{57}{9}\right] + \left[\frac{57}{27}\right] + \left[\frac{57}{81}\right] = 19 + 6 + 2 + 0 = 27 =\propto_2.$$

It must be checked whether $g(56,3) =\propto_2$. It is found that $g(56,3) = 26 <\propto_2$. Results:

A={57,58, ...}.

apply sub-stage II2. $\mathbf{S}(p_2{}^{\propto_2}) = \mathbf{S}(3^{27}) = \min A = 57$.

i=3. Because $\propto_3 > p_3$, apply sub-stage II1: consider the inequation $g(x, 7) \geq 12$; to determine *A*, the set of solutions of the inequation, it is observed that M=7·12=84>$7^2 = 49$, which involves $\mathbf{S}(7^{12}) < M$. Because $\left[\frac{84}{7}\right] - \left[\frac{84}{49}\right] = 11$ choose *x*=84-11=73. $g(73,7) = 11 < 12 =\propto_3$ is obtained, which involves increasing the value of *x*. Choose *x*=77 and $g(77,7) = 12 =\propto_3$ is obtained. It must be checked whether $g(76,7) =\propto_3$. It is found that $g(76,7) = 11 <\propto_3$. Results:

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$$A=\{77,78,\ldots\}.$$

apply sub-stage II2. $\mathbf{S}({p_3}^{\propto_3}) = \mathbf{S}(7^{12}) = \min A = 77.$

Step III. $\mathbf{S}(a) = \max\{\mathbf{S}({p_i}^{\propto_i}) | i = \overline{1,3}\} = \max\{96,57,77\} = 96.$

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