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THE EXTENDED SUM OF THE DIGITS OF A NATURAL NUMBER IN THE DECIMAL REPRESENTATION "THE TEST WITH 9" AND FERMAT'S DIAGNOSIS

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***Abstract**: This note presents a justification of the well-known procedure called "the test with 9" to verify the correctness of some sums or multiplications with natural numbers. The diagnosis of a 12-digit number that was non-prime, was quickly established by Fermat in the 17th century. To verify this result was used "the test with 9".*

***Key words**: algebraic structures: group, unitary commutative ring, the grammar of a language, the divisibility criterion with 9 in ℕ, the relation of congruence.*

***JEL Classification**: C12, C14*

1. EXTENDED SUM

The extended sum of digits of a natural number a in the decimal representation, denoted by $\bar{s}(a)$, is equal to the sum of digits from a if that is at most 9 or otherwise, it is continued with the sum of the digits from the first sum, then with the sum of the digits from the result of the next sum and the sum of the digits is continued until the sum of their is at most nine.

So the extended sum of the digits of a number in the decimal representation is between 0 and 9, that is:

$$\bar{s}: \mathbb{N} \to \{0,1, \dots ,9\}.$$

For example, $\bar{s}(177788) = 2$, because is obtained:

$1 + 3 \cdot 7 + 2 \cdot 8 = 38$ and $38> 9$, then $3 + 8 = 11$ and $11> 9$, then $1 + 1 = 2 <9$, stop.

We define $\bar{s}^{-1}: \{0,1,2, \dots ,9\} \to \mathcal{P}(\mathbb{N})$, where $\bar{s}^{-1}(k) = \{n \in \mathbb{N} | \bar{s}(n) = k, k = \overline{0,9}\}$ and it is said that the natural number n in the decimal representation is in class $\hat{k}$, if $\bar{s}(n) = k$. Classes are obtained $\hat{0}, \hat{1}, \hat{2}, \hat{3}, \hat{4}, \hat{5}, \hat{6}, \hat{7}, \hat{8}$ and $\hat{9}$. Because $\bar{s}(n) = 0$ if and only if n=0, $\hat{0} = \{0\}$.

Let $\mathbb{N}_9^* = \{\hat{1}, \dots, \hat{9}\}$. A ring structure over this set is defined below.

Remark 1. From *the multiplication table* of the natural numbers in the decimal representation, from 1 to 10, we find:

a) for digits 1, 2, 4, 5, 7 and 8, the extended sum of the results from the multiplication table takes all the values in the set {1,2,3,4,5,6,7,8, 9};

b) for digits 3 and 6, the extended sum of the results from the multiplication table takes all the values from the set {3,6,9};

c) for the digit 9, the extended sum of the results from the multiplication table is equal to 9.

Let (L (G), +, ·) be the ring of natural numbers in the decimal representation generated by the left linear grammar G=$(\mathrm{V_N}, \mathrm{V_T}, \mathrm{S}, \mathrm{P})$, [3], where:

$$\mathrm{V_N} = \{\mathrm{S}, \mathrm{T}\}, \mathrm{V_T} = \mathrm{N_{c0}},\ \mathrm{N_{c0}} = \mathrm{N_c} \cup \{0\}, \mathrm{N_c} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}, \text{ and the P productions:} \{\mathrm{S} \to a; a \in \mathrm{V_T}\}, \{\mathrm{S} \to \mathrm{T}a; a \in \mathrm{V_T}\}, \{\mathrm{T} \to \mathrm{T}a; a \in \mathrm{V_T}\} \text{ şi } \{\mathrm{T} \to b; b \in \mathrm{N_c}\}.$$

Any natural number expressed in the ten base is a word above $\mathrm{N_{c0}}$, and $\mathrm{N_{c0}^*}$ is the free monoid generated by the vocabulary $\mathrm{N_{c0}}$ in relation to the concatenation operation.

(If x=$a_{i_1} \dots a_{i_m}, y = b_{j_1} \dots b_{j_p} \in \mathrm{N_{c0}^*}$ then x concatenated with y is denoted by $xy = a_{i_1} \dots a_{i_m} b_{j_1} \dots b_{j_p}$.)

It is found that $\mathbb{N} \subset \mathrm{N_{c0}^*}$, because $\mathrm{N_{c0}^*}$ also contains words starting with 0 (zero).

The following notations will be used:

$$a^{n.} = \underbrace{a \dots a}_{n \text{ concatenations}};$$

$$1\Sigma = \underbrace{1 \dots 1}_{11} = 10^{10} + 10^9 + \dots + 10 + 1 = \sum_{k=0}^{10} 10^k,$$

$$2\Sigma = \underbrace{1 \dots 1}_{1\Sigma} = \sum_{k=0}^{1\Sigma-1} 10^k, \dots, (n+1)\Sigma = \underbrace{1 \dots 1}_{n\Sigma} = \sum_{k=0}^{n\Sigma-1} 10^k, \forall n \geq 1.$$

Also, to write a non-zero number a in the decimal representation, the following two equivalent forms will be used [2]:

$$a=\overline{a_1 \ldots . a_m} \in \mathbb{N}^* \text{ or } a=a_1 \ldots a_m \in \mathbb{N}^*, \text{ where } a_1 \neq 0, a_i \text{ digits}, \forall i=\overline{1,m}.$$

Remark 2. $n\Sigma \in \mathbb{N}$, $\forall n \geq 1$ and

$$(n+1)\Sigma - n\Sigma = \sum_{k=n\Sigma}^{(n+1)\Sigma-1} 10^k = 10^{n\Sigma}\left(10^{(n+1)\Sigma-n\Sigma-1} + 10^{(n+1)\Sigma-n\Sigma-2} + \cdots + 10 + 1\right).$$

Lemma. Let s(a) be the digits sum of the number a, a natural number in the decimal representation. Then:

1) s: $\mathbb{N} \rightarrow \mathbb{N}$ with $|\mathrm{s}(a)| \leq |a|$, for any $a \in \mathbb{N}$, where $|a|$ represents the number of its digits or the length of the word a;

2) Using notation $\mathrm{s}_n = \underbrace{\mathrm{s} \circ \mathrm{s} \circ \ldots \circ \mathrm{s}}_{n}$, for anything $a \in \mathbb{N}$, the sequence $\left(\mathrm{s}_n(a)\right)_{n \in \mathbb{N}}$ is convergent, so that $\lim_{n\to\infty} \mathrm{s}_n(a) = \bar{\mathrm{s}}(a)$. Moreover, the sequence is stationary, because there is $n_a \in \mathbb{N}^*$ depending on m, where m is the order of a, so that:

$$\bar{\mathrm{s}}(a) = \left(\underbrace{\mathrm{s} \circ \mathrm{s} \circ \ldots \circ \mathrm{s}}_{n_a}\right)(a) \text{ and } \mathrm{s}_n(a) = \bar{\mathrm{s}}(a) \text{ for any } n \geq n_a.$$

3) The additivity property of the operator s in relation to the concatenation operation: $\mathrm{s}(a) = \mathrm{s}(\overline{a_1 \ldots a_k}) + \mathrm{s}(\overline{a_{k+1} \ldots a_m})$,

where $a=a_1 \ldots a_k a_{k+1} \ldots a_m \in \mathbb{N}^*$, $a_1 \ldots a_k$ and $a_{k+1} \ldots a_m$ are sub-words of a, $1 \leq k \leq m$, $m \in \mathbb{N}^*$. In particular, if $a_1 = \cdots = a_m = p \neq 0$ is obtained $\mathrm{s}(a) = m \cdot p$.

4) For $a \in \mathbb{N}^*$:

$$\mathrm{s}(a^{n\cdot}) = \mathrm{s}\left(\underbrace{a \ldots a}_{n}\right) = n \cdot \mathrm{s}(a).$$

In particular, if $a \in \mathrm{N_c}$ is obtained $\mathrm{s}(a^{n\cdot}) = n \cdot a$.

Proof.

1) Let $a=a_1 \dots a_m \in \mathbb{N}^*, a_1 \neq 0, a_i \in \mathrm{N}_{c0}, m\in\mathbb{N}^*$.

If m=1, then $a=a_1$ and a it is first order, $|a| = 1=|\mathrm{s}(a)|$ and $\bar{\mathrm{s}}(a) = \mathrm{s}(a) = a_1 \in \mathrm{N}_c$.

If m=2, then $a=a_1a_2$ and a is of the second order, $|a| = 2$. Because $10\leq a\leq 99$, then $a_1 \leq \mathrm{s}(a) \leq a_1 + 9 \leq 18$, $a_1 \leq \bar{\mathrm{s}}(a) = \mathrm{s}(\mathrm{s}(a)) \leq 9$ şi

$$|a| \geq |\mathrm{s}(a)| = \begin{cases} 1, & a_1 \leq \mathrm{s}(a) \leq 9 \\ 2, & 9 < s(a) \leq 18 \end{cases} \geq |\bar{\mathrm{s}}(a)| = 1.$$

....

If m=11, then $a=a_1 \dots a_{11}$ and a is of order 11, $|a| = 11$. Because

$\overline{a_1 \underbrace{0 \dots 0}_{10}} \leq a \leq \underbrace{9 \dots 9}_{11}$, then $a_1 \leq \mathrm{s}(a) \leq 99$, $a_1 \leq \mathrm{s}(\mathrm{s}(a)) \leq 18$, $a_1 \leq \bar{\mathrm{s}}(a) = \mathrm{s}\left(\mathrm{s}(\mathrm{s}(a))\right) \leq 9$ and $|a| > |\mathrm{s}(a)| = \begin{cases} 1, & a_1 \leq s(a) \leq 9 \\ 2, & 9 < s(a) \leq 99 \end{cases} \geq \left|\mathrm{s}(\mathrm{s}(a))\right| = \begin{cases} 1, & a_1 \leq \mathrm{s}(\mathrm{s}(a)) \leq 9 \\ 2, & 9 < s(\mathrm{s}(a)) \leq 18 \end{cases} \geq |\bar{\mathrm{s}}(a)| = 1$.

....

We will prove by mathematical induction with respect to the order m, where $m=n\Sigma$, $n\in\mathbb{N}^*$, that $|s(a)| \leq |a|$.

If $m=1\Sigma = \underbrace{1 \dots 1}_{11}$, then $a=a_1 \dots a_{\underbrace{1 \dots 1}_{11}}$ is of order $\underbrace{1 \dots 1}_{11}$, $|a| = \underbrace{1 \dots 1}_{11}$. Because

$\overline{a_1 \underbrace{0 \dots 0}_{\underbrace{1 \dots 1}_{10}}} \leq a \leq \underbrace{9 \dots 9}_{\underbrace{1 \dots 1}_{11}}$, then $a_1 \leq \mathrm{s}(a) \leq \underbrace{9 \dots 9}_{11}$, $a_1 \leq \mathrm{s}(\mathrm{s}(a)) \leq 99$, $a_1 \leq \mathrm{s}\left(\mathrm{s}(\mathrm{s}(a))\right) \leq 18$, $a_1 \leq \bar{\mathrm{s}}(a) = \mathrm{s}\left(\mathrm{s}\left(\mathrm{s}(\mathrm{s}(a))\right)\right) \leq 9$ and

$$|a| \geq |\mathrm{s}(a)| = \begin{cases} 1, & a_1 \leq \mathrm{s}(a) \leq 9 \\ 2, & 9 < s(a) \leq 99 \\ \dots & \dots \\ 11, & \underbrace{9 \dots 9}_{10} < s(a) \leq \underbrace{9 \dots 9}_{11} \end{cases} \geq \left|\mathrm{s}(\mathrm{s}(a))\right| = \begin{cases} 1, & a_1 \leq \mathrm{s}(\mathrm{s}(a)) \leq 9 \\ 2, & 9 < s(\mathrm{s}(a)) \leq 99 \end{cases} \geq$$

$$\geq \left|s\left(s(s(a))\right)\right| = \begin{cases} 1, & a_1 \leq s\left(s(s(a))\right) \leq 9 \\ 2, & 9 < s\left(s(s(a))\right) \leq 18 \end{cases} \geq \left|\overline{s(a)} = s\left(s\left(s(s(a))\right)\right)\right| = 1.$$

Suppose that the relation is true for any m less than or equal to $(n-1)\Sigma$ and and we will prove the relation for m=$n\Sigma$.

If $m = \underbrace{1 \dots 1}_{n\Sigma}$, then a=$\overline{a_1 \dots a_{n\Sigma}}$ and is of order $n\Sigma$, $|a| = n\Sigma$, ∀n≥1. Because

$\overline{a_1 \quad \underbrace{0 \dots 0}_{n\Sigma - 10^{(n-1)\Sigma - 1}}} \leq a \leq \overline{\underbrace{9 \dots 9}_{n\Sigma}}$ then $a_1 \leq s(a) \leq n\Sigma \cdot 9$, $a_1 \leq s(s(a)) \leq (n-1)\Sigma \cdot 9$, … $a_1 \leq \underbrace{s(\dots(s(a))\dots)}_{n\Sigma} \leq 1\Sigma \cdot 9$, … , $a_1 \leq s\left(\underbrace{s(\dots(s(a))\dots)}_{n\Sigma}\right) \leq \underbrace{9 \dots 9}_{11}$, $a_1 \leq s\left(s\left(\underbrace{s(\dots(s(a))\dots)}_{n\Sigma}\right)\right) \leq 99$, $a_1 \leq s\left(s\left(s\left(\underbrace{s(\dots(s(a))\dots)}_{n\Sigma}\right)\right)\right) \leq 18$, $a_1 \leq \bar{s}(a) =$

$$s\left(s\left(s\left(s\left(\underbrace{s(\dots(s(a))\dots)}_{n\Sigma}\right)\right)\right)\right) \leq 9$$

and

$$|a_1| \geq |s(a)| = \begin{cases} 1, & a_1 \leq s(a) \leq 9 \\ \dots & \dots \\ 11, & \underbrace{9 \dots 9}_{10} < s(a) \leq \underbrace{9 \dots 9}_{11} \\ \dots & \dots \\ (n-1)\Sigma, & \underbrace{9 \dots 9}_{(n-1)\Sigma - 1} < s(a) \leq \underbrace{9 \dots 9}_{(n-1)\Sigma} \end{cases} \geq \left|s(s(a))\right| \geq \dots \geq$$

$$\geq \left|\underbrace{s(\dots(s(a))\dots)}_{n\Sigma+3}\right| = \begin{cases} 1, & a_1 \leq \underbrace{s(\dots(s(a))\dots)}_{n\Sigma+3} \leq 9 \\ 2, & 9 < \underbrace{s(\dots(s(a))\dots)}_{n\Sigma+3} \leq 18 \end{cases} \geq$$

$$\geq \left|\bar{s}(a) = \underbrace{s(\dots(s(a))\dots)}_{n\Sigma+4}\right| = 1.$$

Because for a=$a_1 \dots a_p$ with p greater than $(n-1)\Sigma$ and smaller than $(n-1)\Sigma \cdot 9$ we have $|a| = (n-1)\Sigma$, and the relation is true. Moreover, because in the case p greater than $(n-1)\Sigma \cdot 9$ and smaller than $n\Sigma$ we have $|a| = n\Sigma$, the relation is true in this case, as well, according to the demonstration. In conclusion, the relation 1) is true for any m.

2) Using the results from the demonstration from 1), for any $a\epsilon\mathbb{N}^*$, the sequence $\left(s_n(a)\right)_{n\in\mathbb{N}^*}$ is with positive terms natural numbers, delimited by | a | and descendant, that is, convergent. Moreover, the sequence is also stationary, because there is n_a depending on m, where m is the order of a, so that $s_n(a) = \bar{s}(a)$ for any $n \geq n_a$, where:

$$n_a = \begin{cases} 1, & m = 1 \\ 2, & m = \overline{1,10} \\ 3, & m = \overline{11,1\Sigma - 1} \\ 4, & m = \overline{1\Sigma, \underbrace{1\dots1}_{1\Sigma} - 1} \\ \dots & \dots \\ n\Sigma + 4, & m = \overline{\underbrace{1\dots1}_{(n-1)\Sigma}, \underbrace{1\dots1}_{n\Sigma} - 1} \\ (n+1)\Sigma + 4, & m = \overline{\underbrace{1\dots1}_{n\Sigma}, \underbrace{1\dots1}_{(n+1)\Sigma} - 1} \\ \dots & \dots \end{cases}.$$

It is obtained that: $\lim_{n\to\infty} s_n(a) = \bar{s}\,(a)$, $\forall a\in\mathbb{N}^*$.

3) This property results from the associativity of the sum from $\mathbb{N}$ and from the associativity of the concatenation from $L(G) = \mathbb{N}$. In particular, if $a_1 = \dots = a_m = p$ is obtained $s(p) = p$, p being a digit and $s(a) = \sum_{i=1}^{m} p = m \cdot p$.

4) Equality is deduced by mathematical induction with respect to *n*. For *n* = 2, it is obtain $\mathrm{s}(a^{2\cdot}) = \mathrm{s}(aa) = \mathrm{s}(a) + \mathrm{s}(a) = 2 \cdot \mathrm{s}(a)$, according to the additivity property, and equality is verified. Suppose the equality is true for all natural numbers less than or equal to *n* and we will prove it, based on the assumption, for *n* + 1:

$$\mathrm{s}(a^{n+1\cdot}) = \mathrm{s}(aa^{n\cdot}) = \mathrm{s}(a) + \mathrm{s}(a^{n\cdot}) = \mathrm{s}(a) + \sum_{i=1}^{n} \mathrm{s}(a) = \sum_{i=1}^{n+1} \mathrm{s}(a)$$

and equality is true for any *n.*

The particular case results from $\mathrm{s}(a) = a$.

2. PROPERTIES OF THE EXTENDED SUM

Remark 3. i) There are *a* and *b*, natural numbers in the decimal representation, so that:

$\mathrm{s}(a), \mathrm{s}(b) \neq \mathcal{M}9$, iar $\mathrm{s}(a+b) = \mathcal{M}9$, where $\mathcal{M}9$ means a multiple of 9.

For example, either *a*=123 and *b*=777. Then $\mathrm{s}(a) = 6 \neq \mathcal{M}9, \mathrm{s}(b) = 21 \neq \mathcal{M}9$, but *a*+*b*=900 and $\mathrm{s}(a+b) = 9$=$\mathcal{M}9$.

ii) There are *a* and *b*, natural numbers in the decimal representation, so that:

$\mathrm{s}(a), \mathrm{s}(b) = \mathcal{M}9$, but $\mathrm{s}(a+b) \neq \mathrm{s}(a) + \mathrm{s}(b)$.

For example, either *a*=1107 and *b*=6993. Then $\mathrm{s}(a) = 9 = \mathcal{M}9$, $\mathrm{s}(b) = 27 = \mathcal{M}9$, but *a*+*b*=8100 and $\mathrm{s}(a+b) = 9 \neq 36 = \mathrm{s}(a) + \mathrm{s}(b)$.

Proposition. For any *a*,*b* $\in\mathbb{N}^*$, in the decimal representation, are true the relations:

i) $\bar{\mathrm{s}}(a)$=$\mathrm{s}(a)$mod9, if s $(a)\neq\mathcal{M}9$, $\bar{\mathrm{s}}(a)$=9 if $\mathrm{s}(a)$=$\mathcal{M}9$ and *a*≠0, but $\bar{\mathrm{s}}(a) = 0$ if $a = 0$;

ii) $\bar{\mathrm{s}}(\bar{\mathrm{s}}(a)) = \bar{\mathrm{s}}(a)$;

iii) $\bar{s}(\bar{s}(a)+\mathcal{M}9)=\bar{s}(a)$;

iv)

- 1) The triplet $(\mathbb{N}_9^*,+,\cdot)$ is a commutative and unitary ring, with divisors of the zero, where by definition:

$$\hat{a}+\hat{b}=\widehat{a+b}, \hat{a}\cdot\hat{b}=\widehat{a\cdot b}.$$

- 2) For any $\hat{a} \in \mathbb{N}_9^*$:

$$\hat{a}=\{b\in\mathbb{N}^* \mid \lim_{n\to\infty} s_n\,(b)=\bar{s}(b)=a\} \text{ or}$$
$$\hat{a}=\{b\in\mathbb{N}^* \mid s(b)\text{mod}9=a \text{ and } a\neq 9 \text{ or } b=\mathcal{M}9 \text{ and } a=9\}.$$

- 3) We define $\bar{s}(\hat{a})=\left(\widehat{\bar{s}(b)}\right)$ any would be b a representative of class $\hat{a}$. Then $\bar{s}(\hat{a})=\hat{a}$.

- 4) $\hat{3}^n=\left(\widehat{3^n}\right)=\begin{cases}\hat{3}, & n=1\\ \hat{9}, & n\neq 1\end{cases}, \hat{6}^n=\begin{cases}\hat{6}, & n=1\\ \hat{9}, & n\neq 1\end{cases}$ şi $\hat{9}^n=\hat{9}, \forall n\in\mathbb{N}^*$.

v) $\bar{s}(a+b)=\bar{s}\big(\bar{s}(a)+\bar{s}(b)\big)$;

vi) $\bar{s}(a\cdot b)=\bar{s}(\bar{s}(a)\cdot\bar{s}(b))$;

vii) $\bar{s}(a^n)=\bar{s}\left(\left(\bar{s}(a)\right)^n\right)$;

viii) $\hat{9}=\{a\in\mathbb{N}^* \mid a=\mathcal{M}9\}$.

Proof.

i) Results from Remark 1 (c) and observing that one can write s(a)=$\mathcal{M}$9+r, with $0\leq r\leq 8$.

ii) Results from $0\leq\bar{s}(a)\leq 9$, $s(\bar{s}(a))=\bar{s}(a)$ and

$$\bar{s}\big(\bar{s}(a)\big)=\begin{cases}9=\bar{s}(a), & \text{dacă } \bar{s}(a)=9\\ \bar{s}(a)\text{mod}9=\bar{s}(a), & \text{dacă } \bar{s}(a)\neq 9.\\ 0, & \text{dacă } \bar{s}(a)=0\end{cases}$$

iii) Let $r=\bar{s}(a)$, with $0\leq r\leq 9$. As s($\mathcal{M}$9)=$\mathcal{M}$9, based on the condition of divisibility with 9, and s(r)=r results:

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$$\bar{s}(\bar{s}(a) + \mathcal{M}9) = s((\bar{s}(a) + \mathcal{M}9))\text{mod}9 = s(r + \mathcal{M}9)\text{mod}9 =$$

$$= s(r)\text{mod}9 + s(\mathcal{M}9)\text{mod}9 = r + 0 = \bar{s}(a).$$

iv)

- 1) From the "+" table it follows that $(\mathbb{N}_9^*, +)$ is an abelian group, with the neutral element $e_+ = \hat{9}$ and $-\hat{k} = \widehat{9-k}, k = \overline{1,8}$ and $-\hat{9} = \hat{9}$.

From the "multiply" table it follows that $(\mathbb{N}_9^*, \cdot)$ is commutative monoid with the neutral element $e_\cdot = \hat{1}$. In addition, for any $a,b,c \in \mathbb{N}_9^*$ $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$ and $(\mathbb{N}_9^*, +, \cdot)$ is a unitary commutative ring. Moreover, this ring is with divisors of the zero because $\hat{3} \cdot \hat{6} = \hat{9}$. Class $\hat{9}$ is *absorbent*, i.e. for any $a \in \mathbb{N}_9^*$, $a \cdot \hat{9} = \hat{9}$.

- 2) The first equality results, based on the lemma point 2), and the second equality from i).

- 3) Let $b \in \hat{a}$. Then $\bar{s}(b) = a$ and $\left(\widehat{\bar{s}(b)}\right) = \widehat{(a)} = \hat{a}$, from where equality results. It can be observed that b is some element of class $\hat{a}$, and can even be a.

- 4) By defining $\hat{a}^n = \underbrace{\hat{a} \cdot \ldots \cdot \hat{a}}_{n} = \widehat{a^n}, \forall n \in \mathbb{N}^*$ and equalities are verified by calculation.

v) As $\hat{a} + \hat{b} = \widehat{a+b}$, the equality $\bar{s}\left(\hat{a} + \hat{b}\right) = \bar{s}\left(\widehat{a+b}\right)$ is obtained, and based on iv 3) one can write:

$$\bar{s}\left(\hat{a} + \hat{b}\right) = \widehat{a+b}. \qquad (*)$$

In particular, $\bar{s}(a) = a$, $\bar{s}(b) = b$ şi $\widehat{a+b} = \bar{s}(\widehat{a+b})$. Then from the relation (*) results:

$$\bar{s}\left(\bar{s}(a) + \bar{s}(b)\right) = \bar{s}(a+b).$$

vi) As $\hat{a} \cdot \hat{b} = \widehat{a \cdot b}$, the equality $\bar{s}\left(\hat{a} \cdot \hat{b}\right) = \bar{s}\left(\widehat{a \cdot b}\right)$ is obtained, and based on iv 3) one can write:

$$\bar{s}\left(\hat{a} \cdot \hat{b}\right) = \widehat{a \cdot b}. \qquad (**)$$

In particular, $\bar{s}(a) = a$, $\bar{s}(b) = b$ and $\widehat{a \cdot b} = \bar{s}(\widehat{a \cdot b})$. Then from the relation (**) results:

$$\bar{s}(\bar{s}(a)\cdot\bar{s}(b)) = \bar{s}(a\cdot b).$$

vii) As $(\hat{a})^n = \widehat{a^n}, \forall n \in \mathbb{N}^*$, the equality $\bar{s}((\hat{a})^n) = \bar{s}(\widehat{a^n})$, $\forall n\in\mathbb{N}^*$ is obtained, and based on iv 3) one can write:

$$\bar{s}((\hat{a})^n) = \widehat{a^n}, \forall n \in \mathbb{N}^*. \qquad (***)$$

In particular, $\bar{s}(a) = a$ and $\widehat{a^n} = \widehat{\bar{s}(a^n)}, \forall n \in \mathbb{N}^*$. Then from the relation (***) results:

$$\bar{s}\left(\left(\bar{s}(a)\right)^n\right) = \bar{s}(a^n), \forall n \in \mathbb{N}^*.$$

viii) „⊃” Let $x\in\{a \in \mathbb{N}^* | \ a = \mathcal{M}9\}$. Then $x=\mathcal{M}9 \Leftrightarrow s(x)\vdots 9$, then $s(x) = \mathcal{M}9 \Leftrightarrow s\big(s(x)\big) \vdots 9$, then $s\big(s(x)\big) = \mathcal{M}9$ and so on. Based on point 2) of the lemma there is $n_x \in \mathbb{N}^*$, depending on the order of *x*, so $\bar{s}(x) = \left(\underbrace{s\circ s\circ \ldots \circ s}_{n_x}\right)(x)$. Then $\bar{s}(x) = \mathcal{M}9 \cap \{1,2,\ldots,9\} = 9$ and $x\in\hat{9}$ are obtained based on P2 i). Then $\hat{9} \supset \{a \in \mathbb{N}^* | \ a = \mathcal{M}9\}$.

„⊂” Let $x\in\hat{9}$. Then $x\neq0$, $\bar{s}(x) = 9$ and $\bar{s}(x) = \mathcal{M}9$, hence, based on the lemma point 2) there is $n_x \in \mathbb{N}^*$, depending on the order of *x*, so $\bar{s}(x) = \left(\underbrace{s\circ s\circ \ldots \circ s}_{n_x}\right)(x) = \mathcal{M}9 \Leftrightarrow \left(\underbrace{s\circ s\circ \ldots \circ s}_{n_x-1}\right)(x) = \mathcal{M}9 \Leftrightarrow \left(\underbrace{s\circ s\circ \ldots \circ s}_{n_x-2}\right)(x) = \mathcal{M}9 \Leftrightarrow \cdots \Leftrightarrow s(x) = \mathcal{M}9 \Leftrightarrow x = \mathcal{M}9$. Then $x\in\{a \in \mathbb{N}^* | \ a = \mathcal{M}9\}$ şi $\hat{9} \subset \{a \in \mathbb{N}^* | \ a = \mathcal{M}9\}$. (The chain of equivalences results from the condition of divisibility by 9.)

Consequence. a) "The test with 9" for the sum. In order to verify the correctness of sum *a* with *b*, natural numbers in the decimal representation, it is checked whether the relation in *Proposition* v) is fulfilled; if it is not fulfilled, the sum obtained is not correct. (This is a necessary but not a sufficient condition.)

b) "The test with 9" for the product. In order to verify the correctness of product *a* with *b*, natural numbers in the decimal representation, it is checked whether the relation in *Proposition* vi) is fulfilled; if it is not fulfilled, the product obtained is not correct. (This is a necessary but not a sufficient condition).

3. EXAMPLE

i) Is the calculation correct 222+389=701? By applying the test with 9 is obtained: $\bar{s}(222) = 6, \bar{s}(389) = s(389)\mathrm{mod}9 = 20\mathrm{mod}9 = 2$ and $\bar{s}(\bar{s}(222) + \bar{s}(389)) = 8$, iar $\bar{s}(701) = s(701) = 8$. It would seem that this calculation is correct, in fact the correct result is 611. So passing "the test with 9" is a necessary but not and a sufficient condition for a sum to be correct.

ii) Is the calculation correct 222·389=95358? By applying the test with 9 is obtained: $\bar{s}(222) = 6$, $\bar{s}(389) = s(389)\mathrm{mod}9 = 20\mathrm{mod}9 = 2$ and $\bar{s}(\bar{s}(222) \cdot \bar{s}(389)) = 3$, but $\bar{s}(95358) = s(95358)\mathrm{mod}9 = 30\mathrm{mod}9 = 3$. It would seem that this calculation is correct, in fact the correct result is 86358. So passing "the test with 9 " is a necessary but not and a sufficient condition for a product to be correct.

Application. In [1, pg. 69] the author writes: „it is reported that the French mathematician Pierre Fermat was asked to find out whether the number 100895598169 is a prime number or not. Even the following day, Fermat showed that the respective number is expressed by the product between the numbers 894423 and 112303." To verify this result by applying the test with 9 we obtain: $\bar{s}(894423) = 61\mathrm{mod}9 = 7$, $\bar{s}(112303) = 10\mathrm{mod}9 = 1$, $\bar{s}(7 \cdot 1) = 7$ and $\bar{s}(100895598169) = 61\mathrm{mod}9 = 7$. Since 7 × 1 = 7, it would seem that the result is correct, that is: 100895598169=894423×112303.

By directly calculating the product we get 894423 × 112303 = 100446386169 ≠ 100895598169, the mistake! However, using the software SWP55 we obtain: 100895598169 = 112303 × 898423. So this is a typo mistake, it is need 898423 instead of 894423. Thus, the performance of the mathematician at that time, sec. XVII, is impressive and today.

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