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A SIMULATION OF A HOMOGENEOUS MARKOV CHAIN IN DISCRETE-TIME WHOSE TRANSITION MATRIX CHANGES RANDOMLY

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Abstract: *The paper presents a procedure for simulating a finite and homogeneous Markov chain in discrete time, using random extractions from a finite set of urns. Then the procedure changes to simulate a finite and homogeneous Markov chain in a discrete time whose stochastic transition matrix changes randomly to take into account the context in which stochastic modelling takes place.*

Key Words: *homogeneous Markov chain, simulate, stochastic modeling, stochastic transition matrix, flowchart, Mathematica 5.2.*

JEL Classification: C00, C02.

1. Introduction

First, we present the algorithm for simulating a Markov chain (X (n)), n∈ℕ, finite and homogeneous, [2] and [3], determined by an initial distribution and a stochastic initial transition matrix, both composed of rational numbers. The condition that all probabilities are rational numbers facilitates the use of an urns system to generate the chain. Then, the initial system of urns with which the chain is generated expands by adding an additional urn in order to replace the original stochastic matrix with a new stochastic transition matrix obtained by a transformation that preserves the property of being a stochastic matrix. The time at which the original matrix is replaced is randomly determined using the second additional urn.
Finally, the program in Mathematica 5.2 of the algorithm simulating the finite and homogeneous Markov chain ($\overline{X}$(n)), n∈ℕ, which contains the random change of the original matrix, is indicated.

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2. The simulating a Markov chain (X (n)), n∈N, finite and homogeneous

It is considered $\big(X(n)\big), n \in \mathbb{N}$, a finite and homogeneous Markov chain with the r states, the initial probabilities p (i), i = 1, ..., r and the stochastic transition matrix $\mathbb{P} = (p\ (i, j)) \in \mathcal{M}_r$, $1 \le i, j \le r$.

For $r \in \mathbb{N}^*$, it is considered the set of the urns marked with U_0, U_1, ..., U_r, in each of them, there are at most r types of balls marked with 1,2, ..., r in different or in the same proportions possibly with certain types of missing balls in some urns. Types of balls are also called states.

Let the vector $p = \big(p(i)\big)$, $1 \le i \le r$, where $p(i)$ is the probability of random extraction of a type i ball out of U_0, with return. If there are no balls of some types then there exists $I_0 \subseteq I^r = \{1, \dots, r\}$, with $p(i) > 0$ for $i \in I_0$ or $p(i) = 0$ for $i \notin I_0$ and $\sum_{i=1}^{r} p(i) = 1$.

Let X (0) be the result of an extraction from the urn U_0. X (0) is the random variable defined as follows:

$$X(0): (\Omega_0, \mathcal{P}(\Omega_0), P_0) \to \big(S, \mathcal{P}(S)\big), S = \{1, \dots, r\},$$

$$\Omega_0 = \{\omega_i^0 \big| i \in I^r\}, X(0)(\omega_i^0) = i,$$

and the distribution of X (0) is

$$X(0): \begin{pmatrix} 1 & \cdots & r \\ p(1) & \cdots & p(r) \end{pmatrix},$$

because

$$P_0(X(0) = i) = P_0(\{\omega_i^0\}) = p(i) = \begin{cases} p(i) > 0, & i \in I_0 \\ 0, & i \notin I_0 \end{cases}, \text{ and } \sum_{i=1}^{r} p(i) = 1 .$$

We construct a **procedure** to simulate $n_0 \in \mathbb{N}^*$ terms of $\big(X(n)\big), n \in \mathbb{N}$, using random extractions from the urns marked with U_0, U_1, ..., U_r. The procedure has the following flowchart.

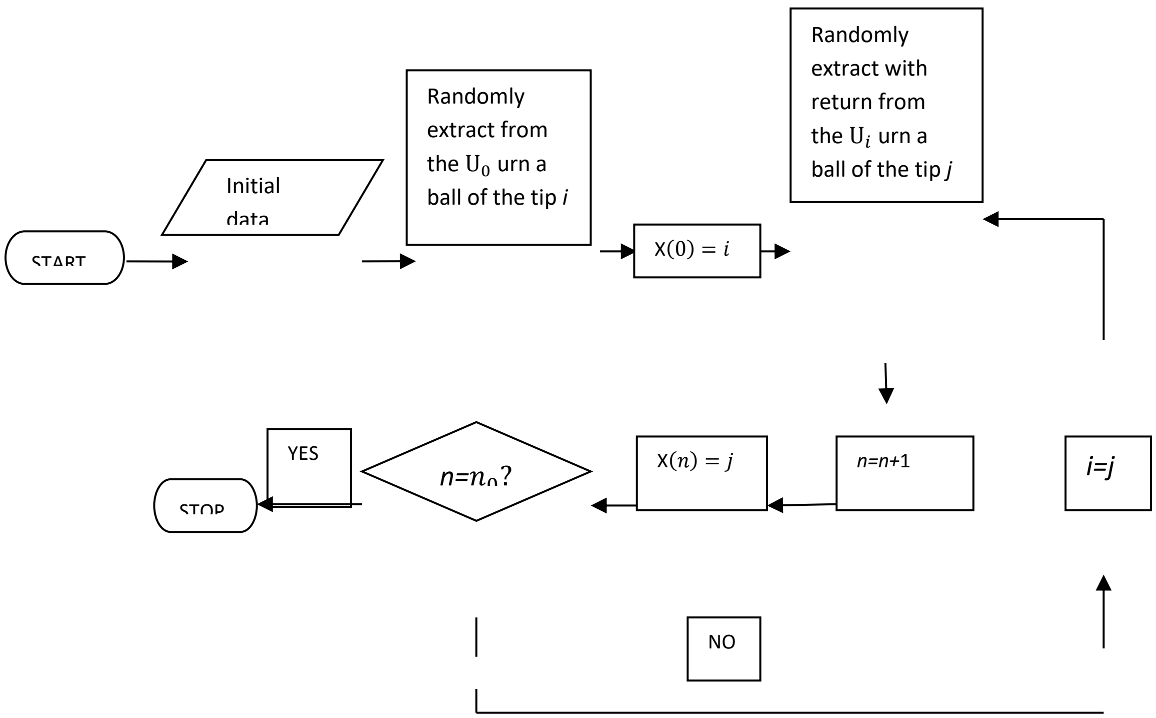

Figure 1. Flowchart of the procedure

Applying this procedure simulates the random variable $X(0)$ in the step 2 and the random variables $X(1)$,..., $X(n_0)$ in the steps 3, 4 and 5.

The *n*-th extraction is also called, suggestively, the step *n*, $n\in \mathbb{N}^*$. Then $X(n)$ represents the state of the *n*-th step.

For 1≤*i*,*j*≤*r*, we consider that $p(i,j)$ is the probability of extraction a *j*-type ball from the urn U_i.

Obsevation. How $p(i,j) = 0$ if urn U_i contains no balls of type *j*, 1≤*j*≤*r*, there exists $I_i \subseteq I^r$ with $p(i,j) > 0$ for $j\in I_i$ or $p(i,j) = 0$ for $j\notin I_i$ and $\sum_{j=1}^{r} p(i,j) = 1$.

Either the matrix $\mathbb{P}=\big(p(i,j)\big) \in \mathcal{M}_r(\mathbb{Q})$. Since $\sum_{j=1}^{r} p(i,j) = 1$ for any 1≤*i*≤*r*, $\mathbb{P}$ is a stochastic matrix.

In these conditions the application of the procedure simulates a string of variables (X (n)), $n\in\mathbb{N}$, which can be argued to be a homogeneous finite Markov chain with the states space S = {1, ..., r}, with the initial distribution given by the vector *p* and with the stochastic transition matrix $\mathbb{P}$.

3. A modification of Markov finite and homogeneous chain

Suppose that under certain circumstances, the matrix $\mathbb{P}$ can be replaced by another stochastic matrix.

Thus, either $\mathbb{B} = \left(b(i,j)\right) \in \mathcal{M}_r(\mathbb{Q}), 1 \leq i,j \leq r$, a stochastic matrix that could be the matrix of the finite and homogeneous Markov chain. A method by which matrix $\mathbb{B}$ can replace matrix $\mathbb{P}$ is the following.

Either a U urn, having two types of balls, from which it is extracted randomly with the probability p_a a type 1 ball and with the probability p_b a type 2 ball, which will determine the stochastic transition matrix of the chain $(X(n))$, $n\in\mathbb{N}$, according to the following **rule**:

- if the result of extraction in U is a ball of type 1, then the chain $(X(n))$, $n\in\mathbb{N}$ has the stochastic transition matrix $\mathbb{P}$;
- if the result of extraction in U is a ball of type 2, then the chain $(X(n))$, $n\in\mathbb{N}$ has the stochastic transition matrix $\mathbb{B}$.

Matrix $\mathbb{B}$, as a rule, comes from the matrix $\mathbb{P}$ by applying some operations or transformations. For example, matrix $\mathbb{B}$ can be obtained by permuting some columns of the matrix $\mathbb{P}$.

Another example is built like this. We define the transformation called **the image in the mirror** of the matrix $\mathbb{P}$ and denoted IM:

$$\text{IM}: \mathcal{M}_r(\mathbb{Q}) \rightarrow \mathcal{M}_r(\mathbb{Q}), \text{IM}(\mathbb{P}) = \mathbb{B} = \left(b(i,j)\right),$$

$$b(i,j) = p(i, r+1-j), \forall i,j = \overline{1,r}.$$

The IM operator retains the property of being a stochastic matrix because:

$$\sum_{j=1}^{r} b(i,j) = \sum_{j=1}^{r} p(i, r+1-j) = p(i,r) + p(i,r-1) + \cdots + p(i,2) + p(i,1) = 1, \ \forall i = \overline{1,r}.$$

Also, we can compose the two kinds of transformations in different ways.

The reason for modifying the initial procedure and introducing urn U is that it allows the stochastic modeling of a real-life phenomenon taking into account the circumstances or the socio-economic context in which the phenomenon occurs.

4. Simulation of the finite and homogeneous Markov chain with the change of the stochastic transition matrix at a random moment

Let be the finite and homogeneous Markov chain (*X* (*n*)), *n*∈ℕ, where *X* (*n*) is the random variable representing the type of extracted ball at the *n*-th extraction from the urns $U_1, \ldots, U_r$, according to the procedure presented, and X (0) representing the type of ball drawn from the urn U_0.

This chain has the space of states S = {1, ..., r}, the initial distribution given by the vector $p=\big(p(i)\big), i = \overline{1,r}$ and the transition stochastic matrix $\mathbb{P}=\big(p(i,j)\big) \in \mathcal{M}_r(\mathbb{Q})$.

Suppose that in a randomly chosen step, the matrix $\mathbb{P}$ is replaced by another stochastic matrix $\mathbb{B} = \big(b(i,j)\big) \in \mathcal{M}_r(\mathbb{Q}), 1 \leq i,j \leq r.$

A method by which matrix $\mathbb{B}$ can replace the matrix $\mathbb{P}$ at the randomly selected p_n step is the following. Supposing that the chain length is n_0 to determine the value of p_n is considered U_p urn, having n_0 balls of the same type, numbered from 1 to n_0. Then a ball is randomly extracted from the urn U_p, and its value is p_n. At the step p_n a ball is randomly extracted from the urn U and the rule in the preceding paragraph applies.

Now, we consider **the modified chain** $\big(\overline{\boldsymbol{X}}(\boldsymbol{n})\big), \boldsymbol{n} \in \mathbb{N}$, obtained from the chain (*X* (*n*)), *n*∈ℕ, which has the states space *S* = {1, ..., *r*}, the initial distribution given by the vector vector $p=\big(p(i)\big), i = \overline{1,r}$ and the transition stochastic matrix $\mathbb{M}=\big(m(i,j)\big) \in \mathcal{M}_r(\mathbb{Q}), 1 \leq i,j \leq r.$

We assume that n_0 random extractions are made from the urns $U_1, \ldots, U_r$, according to the procedure presented. Then the step p_n and the matrix $\mathbb{M}$ are determined as follows.

If ball k is randomly extracted from U_p, then $p_n = k$. At step p_n it will randomly extract a ball from the U urn. By applying the rule above it is obtained:

$$\mathbb{M}=\begin{cases} \mathbb{P}, & \text{if the extracted ball has the color 1} \\ \mathbb{B}, & \text{if the extracted ball has the color 2} \end{cases}.$$

The modified chain simulation algorithm $\big(\overline{\boldsymbol{X}}(\boldsymbol{n})\big), \boldsymbol{n} \in \mathbb{N}$, by performing n_0 random extractions is the following.

The simulation algorithm

* Extract a ball from the U_p *

1. Generate $Up \in U(0,1)$;

2. $p_n = 1 + [n_0 * Up]$;

* Extract a ball from the U_0 urn and it is determinate $\bar{X}(0)$ *

3. Generate $U_0 \in U(0,1)$;

4. If $U_0 \leq p(1)$, then L=1, $\bar{X}(0) = L$ and go to step 6, else go to step 5;

5. If $U_0 > \sum_{i=1}^{L} p(i)$, then $L = L + 1$ and go back to step 5, else $\bar{X}(0) = L$ and go to step 6;

6. n=1;

*A Markov chain is generated with the length of p_n *

7. If $p_n = 1$, then go to step 14, else go to step 8;
8. Generate $UL \in U(0,1)$;
9. If $UL \leq p(\bar{X}(0), 1)$, then J =1, $\bar{X}(n)$=J and go to step 11, else go to step 10;
10. If $UL > \sum_{i=1}^{J} p(\bar{X}(0), i)$, then $J = J + 1$ and go back to step 10, else $\bar{X}(n) = J$;
11. Generate $UL1 \in U(0,1)$;
12. If $UL1 \leq p(\bar{X}(n), 1)$, then k=1, n =n+1, $\bar{X}(n)$=k and go to step 14, else go to step 13;
13. If $UL1 > \sum_{i=1}^{k} p(\bar{X}(n), i)$, then k =k+1 and go back to step 13, else n=n+1 and $\bar{X}(n) = k$;

14. If $n < p_n$, then go back to step 11, else go to step 15;

The stochastic transition matric $\mathbb{M}$ is determined by extracting a ball from the U urn
15. Generate $U \in U(0,1)$;
16. If $U \leq pa$, then $\mathbb{M}=\mathbb{P}$, else $\mathbb{M}=\mathbb{B}$;
A Markov chain is generated with the length of $n_0 - p_n + 1$
17. If $p_n = 1$, then n =$p_n - 1$, else n =p_n;
18. Generate $UL2 \in U(0,1)$;
19. If $UL2 \leq m(\bar{X}(n), 1)$, then k =1, $n = n + 1$, $\bar{X}(n)$=k and go to step 21, else go to step 20;
20. If $UL2 > \sum_{i=1}^{k} m(\bar{X}(n), i)$, then $k = k + 1$ and go back to step 20, else n=n+1 and $\bar{X}(n) = k$;

21. If $n < n_0$, then go back to step 18, else STOP.

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5. A numeric application

We consider the random trajectory [2] of a particle on an oriented axis, it can only occupy the absolute abscissa points 0,1,2,3,4,5,6,7, so that at each moment the *n* particle makes a jump with a unit, to the left with the probability q or to the right with the probability p, $p + q = 1$, $p, q \in (0,1)$, $p, q \in \mathbb{Q}$, and if the particle reaches 0 or 7 is reflected in 1 , respectively in 6 where it remains immobile. The stochastic model of random trajectory is a finite and homogeneous Markov chain $(X(n))$, $n \in \mathbb{N}$, with the states space $S = \{0,1,2,3,4,5,6,7\}$ and with following stochastic transition matrix:

$$\mathbb{P}=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

In order to generate this chain, we consider eight $U_0, \ldots, U_7$ urns in which there are balls of types 0, ..., 7 not necessarily of all types. Thus, in the urn U_0 there are balls of the type 1, in the urn U_1 there are balls of the types 0 and 2, in the urn U_2 balls of types 1 and 3, in the urn U_3 balls of types 2 and 4, in the urn U_4 balls of types 3 and 5, in the urn U_5 balls of types 4 and 6, in the urn U_6 balls of types 5 and 7, and in the urn U_7 there are balls of type 6.

The structure of $U_0, \ldots, U_7$ urns is such that the following relationships take place:

- the probability of extracting a ball of type 0, 1, 2, 3, 4, and 5, respectively, from these urns, is equal to q;

- the probability of extracting a ball of type 2, 3, 4, 5, 6, and 7, respectively, from these urns, is equal to p.

Now let us assume that the random trajectory of a particle takes place in an environment that influences it, the influence consist in the replacement, at a randomly chosen step, of the stochastic transition matrix $\mathbb{P}$ through the matrix $\mathrm{IM}(\mathbb{P}) = \mathbb{B}$, in which case the chain $(X(n))$, $n \in \mathbb{N}$, becomes the chain $(\bar{X}(n)), n \in \mathbb{N}$, whose simulation is accomplished by applying PML1 algorithm.

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The influence of the environment is captured by considering alongside the eight urns, in addition, of a urn U having two types of balls, from which it is randomly extracted with the probability p_a a type 1 ball and with the probability 1-p_a type 2 ball.

In accordance with the rule in paragraph 3, with the help of a random extraction from urn U it is determined whether matrix B replaces the matrix $\mathbb{P}$.

For p = 0.6, q = 0.4, p_a= 0.8 and the initial probabilities p (i), i = 1, ..., 8 given by the vector po = (0,1,0,0,0,0,0,0) the chain simulation algorithm $\left(\bar{X}(n)\right), n \in \mathbb{N}$, with the length n_0 has the following program in Mathematica 5.2:

```
Block[{p={{0,1,0,0,0,0,0,0},{0.4,0,0.6,0,0,0,0,0},{0,0.4,0,0.6,0,0,0,0},{0,0,0.4,0,0.6,0,0,0},{0,0,0,0.4,0,0.6,0,0},{0,0,0,0,0.4,0,0.6,0},{0,0,0,0,0,0.4,0,0.6},{0,0,0,0,0,0,1,0}},

po={0,1,0,0,0,0,0,0},pa=0.8,r=9,no,k=1,J=2},

MapIndexed[xbar[#1,First[#2]]&,Table[1,{21}]]; b=Reverse/@p;Print["b=",{b}];

up=Random[];pn=1+Floor[no×up];Print["pn=",{pn}];

If[k<r,uo=Ranom[];Print["uo=',{uo}'];If[uo<po[[1]],xbar[1,1]=k,

While[uo>∑_{i=1}^{k} po[[i]] , k + +];xbar[1,1]=k],k=r];Print["xbar=",{xbar}];

If[pn≤1,u1=Random[];Print["u1=",{u1}];If[u1≤pa,m=p,m=b];Print["m=",{m}];no=no+1,

Do[If[J<r,u2=Random[];Print["u2=",{u2}];If[u2≤m[[xbar[1,1],1]],xbar[1,n]=J,

While[u2>∑_{i=1}^{J} m[[xbar[1,1], i]],J++];xbar[1,n]=J],u3=Random[];Print["u3=",{u3}];

If[u3≤m[[xbar[1,n],1]],xbar[1,n]=J,While[u3>∑_{i=1}^{J} m[[xbar[1, n], i]],J++];xbar[1,n]=J]];

Print[xbar[1,n]];xbar[1,1]=xbar[1,n],{n,2,no,1}],

Do[If[J<r,u2=Random[];Print["u2=",{u2};If[u2≤p[[xbar[1,1],1]],xbar[1,n]=J,

While[u2>∑_{i=1}^{J} p[[xbar[1, n], i]],J++];xbar[1,n]=J]];

Print[xbar[1,n]];xbar[1,1]=xbar[1,n],{n,2,pn,1}];u1=Random[];Print["u1=",{u1}];

If[u1≤pa,m=p,m=b];Print["m=",{m}];nn=pn+1;J=2;
```

```
Do[If[J<r, u4=Random[];Print["u4=",{u4}];If[u4≤m[[xbar[1,pn],1]],xbar[1,n]=J]];
Print[xbar[1,n]];xbar[1,pn]=xbar[1,n],{n,nn,no,1}]]]
```

The laptop used for the simulations has Intel Pentium (R) Dual-Core based, running at 2.10GHz, for which a simulation time was under 1 minute. The following table shows the simulation results of 400 of Markov chains for $n_0 \in \{20,\ 30,\ 40\}$.

TABLE 1

m	Last state before of the moment Pn	The absorbent state towards which chain converges
p	_ , 1 , 3 , 4 , 5 , 6 , 7	7
b	_ , 1 , 2, 3 , 4, 5 , 6, 7	4 , 5 , 6 , 7

The sign "_" signifies that pn = 1, ie at the time one it is randomly determined if the matrix *p* changes or not.

Conclusion

1. If m = p, irrespective of the last state of the moment pn, the state of absorption towards which the chain converges is state 7. This corresponds to the theoretical conclusion, simulation being adequate.
2. If m = b, depending on the ultimate state before moment pn, the absorbing state towards which the chain converges is one of states 4, 5, 6 or 7. This only partially matches with the theory, simulation being less appropriate.

Bibliography

[1] DOOB, J.L. (1953), Stochastic Processes, New York; John Wiley&Sons; Inc. London; Chapman&Hall, Limited.

[2] IOSIFESCU, M. (1977), Markov chains with a finite number of states and applications (Translation from Romanian), Editura Tehnică, Bucureşti.

[3] Tolver, A. (2016), An introduction to Markov chains, http://web.math.ku.dk/noter/filer/stoknoter.pdf .